Scientific Research Publishing

Applied Mathematics, 2022, 13, 147-162
https://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

Enclosing Ellipses by Folding Disks

Peter Paul Klein

University of Technology Clausthal, Clausthal-Zellerfeld, Germany
Email: peter.p.klein@t-online.de

How to cite this paper: Klein, P.P. (2022) Enclosing Ellipses by Folding Disks. *Applied Mathematics*, **13**, 147-162.
https://doi.org/10.4236/am.2022.132012

Received: December 28, 2021
Accepted: February 18, 2022
Published: February 21, 2022

Abstract

Ellipses can be constructed by folding disks. These folds are forming an envelope of tangents to the ellipse. In the paper of Gorkin and Shaffer, it was shown that the ellipse constructed by folding can be circumscribed by an arbitrary triangle of tangents, the vertices of which are lying on the circumference of the disk. They offered two non-elementary methods of proof, one using Poncelet's Theorem, the other employing Blaschke products. In this paper, it is the intention to present an elementary proof by means of analytic geometry.

Keywords

Straight Line, Perpendicular Bisector, Linear System, Determinant, Point of Intersection, Gardner Ellipse, Bidirectional Folding

1. Introduction

The disk with which the folding is performed shall be represented by a circle $x^2+y^2=r^2$ with radius $r>0$ and center $C=(0,0)$. The point D inside the circle and different from the center C shall be described as $D=(d,0)$ with $|d|<r$. The fold performed by laying a point A_0 of the circumference of the disk on top of D can be described by the perpendicular bisector of the line segment $\overline{A_0D}$, where $A_0=(\xi_0,\eta_0)$ fulfills $\xi_0^2+\eta_0^2=r^2$. The straight line running through $D=(d,0)$ and A_0 has the form

$$\eta_0 x-(\xi_0-d)y=\eta_0 d. \tag{1}$$

The perpendicular bisector of line segment $\overline{A_0D}$ is given by

$$(\xi_0-d)x+\eta_0 y=\frac{1}{2}(r^2-d^2). \tag{2}$$

The point of intersection of both straight lines (1) and (2) is

$M_0=\left(\frac{1}{2}(\xi_0+d),\frac{1}{2}\eta_0\right)$. They are perpendicular to each other because their normal vectors are orthogonal: $\eta_0(\xi_0-d)-(\xi_0-d)\eta_0=0$. See **Figure 1**.

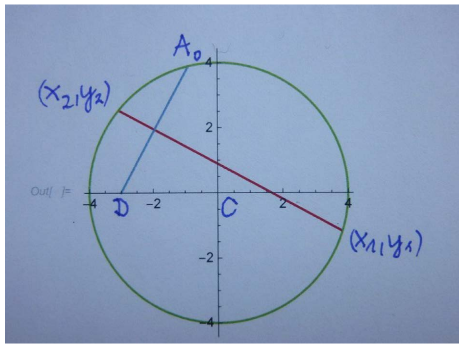

Figure 1. Perpendicular bisector of $\overline{A_0 D}$.

The term "sequential folding" was established by Gorkin and Shaffer [1] and means: After the first fold, proceed in such a way as to start the next fold where the previous fold ended. In contrast to Gorkin and Shaffer [1] in this paper after the first fold, the second fold and the third fold are starting at the two different ends of the first fold on the circle $x^2 + y^2 = r^2$, at (x_1, y_1) and (x_2, y_2) respectively. This shall be named "bidirectional folding". In the second fold, $A_1 = (\xi_1, \eta_1)$ has to be chosen on the circle $x^2 + y^2 = r^2$ in such a way that (x_1, y_1) is a point on the perpendicular bisector of $\overline{A_1 D}$:

$$(\xi_1 - d)x_1 + \eta_1 y_1 = \frac{1}{2}(r^2 - d^2), \quad \xi_1^2 + \eta_1^2 = r^2. \tag{3}$$

In the third fold, a point $A_2 = (\xi_2, \eta_2)$ has to be chosen on the circle $x^2 + y^2 = r^2$ such that (x_2, y_2) is a point on the perpendicular bisector of $\overline{A_2 D}$:

$$(\xi_2 - d)x_2 + \eta_2 y_2 = \frac{1}{2}(r^2 - d^2), \quad \xi_2^2 + \eta_2^2 = r^2. \tag{4}$$

The final problem will be, to show that perpendicular bisectors of $\overline{A_1 D}$ and $\overline{A_2 D}$

$$\begin{aligned} (\xi_1 - d)x + \eta_1 y &= \frac{1}{2}(r^2 - d^2) \\ (\xi_2 - d)x + \eta_2 y &= \frac{1}{2}(r^2 - d^2) \end{aligned} \tag{5}$$

have a point of intersection (x_T, y_T) on the circle $x^2 + y^2 = r^2$. Because of the quadratic Equations in (3) and (4) involved in the calculation of (ξ_1, η_1) and (ξ_2, η_2) we get two solutions $((\xi_1)_\pm, (\eta_1)_\mp)$ and $((\xi_2)_\pm, (\eta_2)_\mp)$ respectively. The number of solutions can be reduced, however, looking at two straight lines: one passing through $((\xi_1)_+, (\eta_1)_-)$ and $((\xi_1)_-, (\eta_1)_+)$ the other through $((\xi_2)_+, (\eta_2)_-)$ and $((\xi_2)_-, (\eta_2)_+)$. It will be shown that the point of intersection (x_S, y_S) of both straight lines is located on the circle $x^2 + y^2 = r^2$ and is equal to $A_0 = (\xi_0, \eta_0)$.

The procedure described above constructs an arbitrary triangle circumscribing an ellipse and having vertices on a circle. This is in contrast to the approach in [1], where a special triangle was chosen to circumscribe an ellipse and have vertices on a circle. The generalization to an arbitrary triangle is accomplished by employing Poncelet's Theorem [2]. In a remark in [1], a different way of proof is indicated with a Blaschke product of degree three, treated more extensively in [3].

2. Preliminaries

Next we want to find the points of intersection of (2) with the circle $x^2+y^2=r^2$. Assuming $\eta_0 \neq 0$ Equation (2) can be rewritten as

$$y=-\frac{\xi_0-d}{\eta_0}x+\frac{r^2-d^2}{2\eta_0} \tag{6}$$

which gives after substitution into the equation of the circle and multiplication with η_0^2

$$\eta_0^2x^2+\left(-(\xi_0-d)x+\frac{1}{2}\left(r^2-d^2\right)\right)^2-\eta_0^2r^2=0.$$

So we get

$$\left((\xi_0-d)^2+\eta_0^2\right)x^2-(\xi_0-d)\left(r^2-d^2\right)x+\frac{1}{4}\left(r^2-d^2\right)^2-\eta_0^2r^2=0. \tag{7}$$

The highest coefficient in (7) is different from zero for (ξ_0,η_0) with $\xi_0^2+\eta_0^2=r^2$ because

$$(\xi_0-d)^2+\eta_0^2=r^2-2\xi_0d+d^2\geq r^2-2r|d|+d^2=\left(r-|d|\right)^2>0. \tag{8}$$

So we get

$$x^2-\frac{(\xi_0-d)\left(r^2-d^2\right)}{(\xi_0-d)^2+\eta_0^2}x+\frac{\frac{1}{4}\left(r^2-d^2\right)^2-\eta_0^2r^2}{(\xi_0-d)^2+\eta_0^2}=0. \tag{9}$$

Before writing down the solutions of (9) we are introducing the expression

$$R(\xi_0,\eta_0)=4r^2\left((\xi_0-d)^2+\eta_0^2\right)-\left(r^2-d^2\right)^2. \tag{10}$$

Lemma 1

$R(\xi_0,\eta_0)$ is positive for (ξ_0,η_0) with $\xi_0^2+\eta_0^2=r^2$.

Proof: The expression $(\xi_0-d)^2+\eta_0^2$ is bounded below according to (8). Thus

$$R(\xi_0,\eta_0)\geq 4r^2\left(r-|d|\right)^2-\left(r-|d|\right)^2\left(r+|d|\right)^2\geq\left(r-|d|\right)^2\left(4r^2-\left(r+|d|\right)^2\right).$$

For the expression $4r^2-\left(r+|d|\right)^2$ holds

$$4r^2-\left(r+|d|\right)^2=\left(2r-\left(r+|d|\right)\right)\left(2r+\left(r+|d|\right)\right)=\left(r+|d|\right)\left(3r+|d|\right)>0.$$

Therefore $R(\xi_0,\eta_0)$ is positive for (ξ_0,η_0) with $\xi_0^2+\eta_0^2=r^2$. □

With expression (10) the solutions of (9) have the form

$$x_{1,2}=\frac{1}{2\left((\xi_0-d)^2+\eta_0^2\right)}\left((\xi_0-d)\left(r^2-d^2\right)\pm\eta_0\sqrt{R(\xi_0,\eta_0)}\right), \tag{11}$$

where the index 1 of x shall correspond to the +sign and the index 2 to the −sign. Substituting (11) into (6) we obtain

$$y_{1,2} = \frac{1}{2\left(\left(\xi_0 - d\right)^2 + \eta_0^2\right)}\left(\eta_0\left(r^2 - d^2\right) \mp \left(\xi_0 - d\right)\sqrt{R\left(\xi_0, \eta_0\right)}\right), \tag{12}$$

where the index 1 of y corresponds to the −sign and the index 2 to the +sign. For $i \in \{1,2\}$ this makes for $x_i^2 + y_i^2 = r^2$.

Although the derivation of (11) was carried out under the assumption $\eta_0 \neq 0$, the case $\eta_0 = 0$ can be recovered from (11). For $\eta_0 = 0$ follows $\xi_0 = \pm r$. For $\xi_0 = +r$ we get $x_1 = \frac{1}{2}(r + d)$ and for $\xi_0 = -r$ we have $x_2 = -\frac{1}{2}(r - d)$. In each case we are getting from $x_{1,2}^2 + y_{1,2}^2 = r^2$ two y-values: $y_1 = \pm\frac{1}{2}\sqrt{(3r + d)(r - d)}$ and $y_2 = \pm\frac{1}{2}\sqrt{(3r - d)(r + d)}$.

From (11) and (12) we can deduce for $i \in \{1,2\}$:

$$\left(\xi_0 - d\right)x_i + \eta_0 y_i = \frac{1}{2}\left(r^2 - d^2\right). \tag{13}$$

3. Construction of the Gardner Ellipse

The term "Gardner ellipse" has been used in [1] for the ellipse constructed by folding disks, going back to Gardner's publication [4]. In order to calculate the point of intersection (x_B, y_B) of the straight line through $C = (0,0)$ and $A_0 = (\xi_0, \eta_0)$ and straight line (2) we are looking at the linear system

$$\begin{aligned} &\eta_0 x_B - \xi_0 y_B = 0 \\ &\left(\xi_0 - d\right)x_B + \eta_0 y_B = \frac{1}{2}\left(r^2 - d^2\right). \end{aligned} \tag{14}$$

For the determinant D_B of the linear system (14) holds:

$$D_B = \eta_0^2 + \left(\xi_0 - d\right)\xi_0 = \eta_0^2 + \xi_0^2 - \xi_0 d = r^2 - \xi_0 d$$

and we have the following positive lower bound

$$D_B = r^2 - \xi_0 d \geq r^2 - |\xi_0||d| \geq r^2 - r|d| = r\left(r - |d|\right) > 0.$$

The solutions of the linear system (14) are

$$x_B = \frac{\xi_0\left(r^2 - d^2\right)}{2\left(r^2 - \xi_0 d\right)}, \quad y_B = \frac{\eta_0\left(r^2 - d^2\right)}{2\left(r^2 - \xi_0 d\right)}. \tag{15}$$

Since $B = (x_B, y_B)$ is lying on straight line (2), the perpendicular bisector of $\overline{A_0 D}$, the line segments $\overline{DB}$ and $\overline{A_0 B}$ have the same length. Thus

$$\overline{DB} + \overline{BC} = \overline{A_0 B} + \overline{BC} = \overline{A_0 C} = r. \tag{16}$$

See **Figure 2**.

Letting point $A_0 = (\xi_0, \eta_0)$ move on circle $x^2 + y^2 = r^2$ we obtain a sequence of points $B = (x_B, y_B)$ which have a constant sum of distances from two fixed points C and D. According to (16) this sum of distances is equal to r.

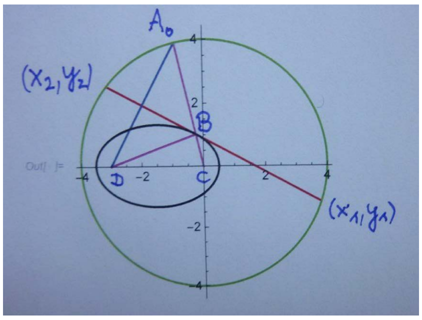

Figure 2. Gardner ellipse with tangent.

Therefore the so generated points are forming an ellipse, the so-called "Gardner ellipse". It has foci C and D and midpoint $\left(\frac{1}{2}d, 0\right)$. The semi axes of the Gardner ellipse are given by $a=\frac{r}{2}$ and $b=\frac{1}{2}\sqrt{r^2-d^2}$. The formula of the ellipse is

$$\frac{\left(x-\frac{d}{2}\right)^2}{a^2}+\frac{y^2}{b^2}=1. \quad (17)$$

The coordinates of $B=(x_B, y_B)$ given in (15) fulfill Equation (17), if the values for the semi axes a and b are substituted.

The tangent to the ellipse (17) at the point (x_B, y_B) has the form

$$\frac{\left(x_B-\frac{d}{2}\right)\left(x-\frac{d}{2}\right)}{a^2}+\frac{y_B y}{b^2}=1. \quad (18)$$

Proposition 1

The tangent to ellipse (17) at the point (x_B, y_B), given by (18), is equivalent to the perpendicular bisector (2) of line segment $\overline{A_0 D}$.

Proof: From (15) we get

$$x_B-\frac{d}{2}=\frac{\xi_0\left(r^2-d^2\right)-d\left(r^2-\xi_0 d\right)}{2\left(r^2-\xi_0 d\right)}=\frac{r^2\left(\xi_0-d\right)}{2\left(r^2-\xi_0 d\right)}$$

and

$$\frac{x_B-\frac{d}{2}}{a^2}=2\frac{\xi_0-d}{r^2-\xi_0 d}, \quad \frac{y_B}{b^2}=2\frac{\eta_0}{r^2-\xi_0 d}.$$

This leads to

$$\frac{\left(x_B-\frac{d}{2}\right)\left(x-\frac{d}{2}\right)}{a^2}+\frac{y_B y}{b^2}=2\frac{\xi_0-d}{r^2-\xi_0 d}\left(x-\frac{d}{2}\right)+2\frac{\eta_0}{r^2-\xi_0 d}y=1,$$

$$\left(\xi_0-d\right)\left(x-\frac{d}{2}\right)+\eta_0 y=\frac{1}{2}\left(r^2-\xi_0 d\right),$$

$$\left(\xi_0-d\right)x+\eta_0 y=\frac{1}{2}\left(r^2-\xi_0 d\right)+\frac{d}{2}\left(\xi_0-d\right)=\frac{1}{2}\left(r^2-d^2\right),$$

which is the perpendicular bisector (2) of line segment $\overline{A_0 D}$.

Remark: Because of the relations for the semi axes of the Gardner ellipse $a=\frac{r}{2}$ and $b=\frac{1}{2}\sqrt{r^2-d^2}$ an arbitrary ellipse with semi axes *a* and *b* can be interpreted as Gardner ellipse, choosing $r=2a$ as radius of the surrounding circle and $d=\sqrt{r^2-(2b)^2}=2\sqrt{a^2-b^2}$ as distance of the foci.

4. Bidirectional Folding

Having obtained the points (x_1,y_1) and (x_2,y_2) according to (11) and (12) the second fold is starting from (x_1,y_1) and the third fold from (x_2,y_2), described by Equations (3) and (4) respectively.

The calculation of the points $A_1=(\xi_1,\eta_1)$ and $A_2=(\xi_2,\eta_2)$ is combined by looking for $A_i=(\xi_i,\eta_i)$ for $i\in\{1,2\}$ with

$$(\xi_i-d)x_i+\eta_i y_i=\frac{1}{2}(r^2-d^2),\ \ \xi_i^2+\eta_i^2=r^2. \tag{19}$$

For $y_i=0$, because of $x_i=\pm r$, we obtain from (19) for $i\in\{1,2\}$

$$\xi_i=d\pm\frac{1}{2r}(r^2-d^2)\ \text{ and }\ \eta_i=\pm\sqrt{r^2-\xi_i^2}.$$

In the case $y_i\neq 0$ we use the expansion

$$\xi_i^2=(\xi_i-d+d)^2=(\xi_i-d)^2+2d(\xi_i-d)+d^2$$

and multiply the second Equation of (19) with y_i^2

$$\left((\xi_i-d)^2+2d(\xi_i-d)+d^2\right)y_i^2+\eta_i^2y_i^2=r^2y_i^2. \tag{20}$$

Solving the first Equation of (19) for $\eta_i y_i$ and substituting in (20) gives

$$(\xi_i-d)^2y_i^2+2dy_i^2(\xi_i-d)+\left(-(\xi_i-d)x_i+\frac{1}{2}(r^2-d^2)\right)^2-(r^2-d^2)y_i^2=0,$$

$$(\xi_i-d)^2(x_i^2+y_i^2)+\left(2dy_i^2-x_i(r^2-d^2)\right)(\xi_i-d)$$
$$+\frac{1}{4}(r^2-d^2)^2-(r^2-d^2)y_i^2=0.$$

Because of $x_i^2+y_i^2=r^2$ we obtain a quadratic equation for ξ_i-d:

$$\begin{aligned}&(\xi_i-d)^2-\frac{2}{r^2}\left(\frac{x_i}{2}(r^2-d^2)-dy_i^2\right)(\xi_i-d)\\&+\frac{1}{r^2}\left[\left(\frac{1}{2}(r^2-d^2)\right)^2-(r^2-d^2)y_i^2\right]=0.\end{aligned} \tag{21}$$

Before writing down the solutions of (21) we are introducing the expression

$$S(x_i)=r^4-\left(dx_i+\frac{1}{2}(r^2-d^2)\right)^2. \tag{22}$$

Lemma 2

For $i\in\{1,2\}$ and $x_i^2+y_i^2=r^2$ the expression $S(x_i)$ is positive.

Proof: For $i \in \{1,2\}$ the expression $S(x_i)$ can be split up in the factors

$$S(x_i) = \left[r^2 - \left(dx_i + \frac{1}{2}(r^2 - d^2)\right)\right]\left[r^2 + \left(dx_i + \frac{1}{2}(r^2 - d^2)\right)\right]$$
$$= \frac{1}{2}(r^2 - 2dx_i + d^2)\frac{1}{2}(2(r^2 + dx_i) + r^2 - d^2).$$

$S(x_i)$ is positive for $x_i^2 + y_i^2 = r^2$ with $i \in \{1,2\}$ since

$$r^2 - 2dx_i + d^2 \geq r^2 - 2|d|r + d^2 = (r - |d|)^2 > 0$$

and

$$2(r^2 + dx_i) + r^2 - d^2 \geq 2r(r - |d|) + r^2 - d^2 > 0.$$

□

With expression (22) the solutions of (21) are

$$(\xi_i - d)_\pm = \frac{1}{r^2}\left[\frac{x_i}{2}(r^2 - d^2) - dy_i^2 \pm y_i\sqrt{S(x_i)}\right]. \tag{23}$$

From (23) we obtain

$$(\xi_i)_\pm = \frac{1}{r^2}\left[x_i\left(dx_i + \frac{1}{2}(r^2 - d^2)\right) \pm y_i\sqrt{S(x_i)}\right]. \tag{24}$$

Substituting (23) into the first Equation of (19) yields

$$(\eta_i)_\mp = \frac{1}{r^2}\left[y_i\left(dx_i + \frac{1}{2}(r^2 - d^2)\right) \mp x_i\sqrt{S(x_i)}\right] \tag{25}$$

or expressed otherwise: (23) and (25) combine to

$$(\xi_i - d)_\pm x_i + (\eta_i)_\mp y_i = \frac{1}{2}(r^2 - d^2). \tag{26}$$

In addition $(\xi_i)_\pm^2 + (\eta_i)_\mp^2 = r^2$ can be verified.

5. Interpretation of Solutions

For $i \in \{1,2\}$ we are looking at the straight lines passing through

$$((\xi_i)_+, (\eta_i)_-) \text{ and } ((\xi_i)_-, (\eta_i)_+). \tag{27}$$

Proposition 2

For $i \in \{1,2\}$ the straight line

$$x_i x + y_i y = x_i(\xi_i)_+ + y_i(\eta_i)_- \tag{28}$$

is passing through points (27).

Proof: It is clear that $((\xi_i)_+, (\eta_i)_-)$ is fulfilling (28). In order to prove

$$x_i(\xi_i)_- + y_i(\eta_i)_+ = x_i(\xi_i)_+ + y_i(\eta_i)_-$$

we are going to show

$$x_i((\xi_i)_- - (\xi_i)_+) + y_i((\eta_i)_+ - (\eta_i)_-) = 0. \tag{29}$$

According to (24) and (25) we have:

$$\begin{aligned}(\xi_i)_- - (\xi_i)_+ &= \frac{1}{r^2}\left[x_i\left(dx_i + \frac{1}{2}(r^2 - d^2)\right) - y_i\sqrt{S(x_i)}\right] \\ &\quad - \frac{1}{r^2}\left[x_i\left(dx_i + \frac{1}{2}(r^2 - d^2)\right) + y_i\sqrt{S(x_i)}\right] \\ &= -\frac{2}{r^2}y_i\sqrt{S(x_i)}, \\ (\eta_i)_+ - (\eta_i)_- &= \frac{1}{r^2}\left[y_i\left(dx_i + \frac{1}{2}(r^2 - d^2)\right) + x_i\sqrt{S(x_i)}\right] \\ &\quad - \frac{1}{r^2}\left[y_i\left(dx_i + \frac{1}{2}(r^2 - d^2)\right) - x_i\sqrt{S(x_i)}\right] \\ &= \frac{2}{r^2}x_i\sqrt{S(x_i)}.\end{aligned}$$

Thus (29) follows from

$$x_i\left((\xi_i)_- - (\xi_i)_+\right) + y_i\left((\eta_i)_+ - (\eta_i)_-\right) = \frac{2}{r^2}\left(-x_i y_i + y_i x_i\right)\sqrt{S(x_i)} = 0.$$

□

Next we are looking at the point of intersection (x_S, y_S) of the straight lines (28), *i.e.* the solution of the following linear system

$$\begin{aligned}x_1 x_S + y_1 y_S &= x_1(\xi_1)_+ + y_1(\eta_1)_- \\ x_2 x_S + y_2 y_S &= x_2(\xi_2)_+ + y_2(\eta_2)_-.\end{aligned} \tag{30}$$

See Figure 3.

Proposition 3

For the the determinant D_S of the linear system (30) holds

$$D_S = x_1 y_2 - x_2 y_1 = \frac{r^2 - d^2}{2\left((\xi_0 - d)^2 + \eta_0^2\right)}\sqrt{R(\xi_0, \eta_0)}, \tag{31}$$

which is different from zero for (ξ_0, η_0) with $\xi_0^2 + \eta_0^2 = r^2$.

Proof: According to (11) and (12) we have

$$x_1 y_2 = \frac{1}{4\left((\xi_0 - d)^2 + \eta_0^2\right)}\left(4r^2(\xi_0 - d)\eta_0 + (r^2 - d^2)\sqrt{R(\xi_0, \eta_0)}\right),$$

$$x_2 y_1 = \frac{1}{4\left((\xi_0 - d)^2 + \eta_0^2\right)}\left(4r^2(\xi_0 - d)\eta_0 - (r^2 - d^2)\sqrt{R(\xi_0, \eta_0)}\right).$$

This gives the result (31).

D_S is different from zero for (ξ_0, η_0) with $\xi_0^2 + \eta_0^2 = r^2$ because by Lemma 1 $R(\xi_0, \eta_0)$ is positive and $r^2 - d^2 > 0$ holds. □

Theorem 1

For $i \in \{1, 2\}$ holds:

$$\text{Either } \left((\xi_i)_+, (\eta_i)_-\right) = (\xi_0, \eta_0) \text{ or } \left((\xi_i)_-, (\eta_i)_+\right) = (\xi_0, \eta_0). \tag{32}$$

Proof: Because of (13) we have for $i \in \{1, 2\}$

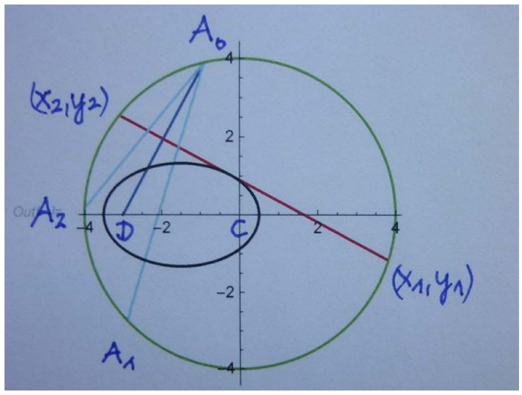

Figure 3. Straight lines through $((\xi_i)_+,(\eta_i)_-)$ and $((\xi_i)_-,(\eta_i)_+)$ for $i\in\{1,2\}$.

$$x_i\xi_0+y_i\eta_0=dx_i+\frac{1}{2}\left(r^2-d^2\right). \tag{33}$$

Furthermore we have for $i\in\{1,2\}$

$$x_i(\xi_i)_++y_i(\eta_i)_-=dx_i+\frac{1}{2}\left(r^2-d^2\right), \tag{34}$$

because (24) and (25) yield

$$\begin{aligned}
&x_i(\xi_i)_++y_i(\eta_i)_-\\
&=\frac{x_i}{r^2}\left[x_i\left(dx_i+\frac{1}{2}\left(r^2-d^2\right)\right)+y_i\sqrt{S(x_i)}\right]\\
&\quad+\frac{y_i}{r^2}\left[y_i\left(dx_i+\frac{1}{2}\left(r^2-d^2\right)\right)-x_i\sqrt{S(x_i)}\right]\\
&=\frac{1}{r^2}\left[\left(x_i^2+y_i^2\right)\left(dx_i+\frac{1}{2}\left(r^2-d^2\right)\right)+\left(x_iy_i-y_ix_i\right)\sqrt{S(x_i)}\right]\\
&=dx_i+\frac{1}{2}\left(r^2-d^2\right).
\end{aligned}$$

Combining (33) and (34) we get for $i\in\{1,2\}$

$$x_i\xi_0+y_i\eta_0=x_i(\xi_i)_++y_i(\eta_i)_-.$$

This means

$$\begin{aligned}
x_1\xi_0+y_1\eta_0&=x_1(\xi_1)_++y_1(\eta_1)_-\\
x_2\xi_0+y_2\eta_0&=x_2(\xi_2)_++y_2(\eta_2)_-.
\end{aligned} \tag{35}$$

According to Proposition 3 the determinant D_S is different from zero for (ξ_0,η_0) with $\xi_0^2+\eta_0^2=r^2$. Therefore the linear systems (30) and (35) have one and only one solution, which leads to $(x_S,y_S)=(\xi_0,\eta_0)$. Point $A_0=(\xi_0,\eta_0)$ is lying on both straight lines (28) and on the circle $x^2+y^2=r^2$. One of the points of intersection of both straight lines (28) with the circle, $((\xi_i)_+,(\eta_i)_-)$ or $((\xi_i)_-,(\eta_i)_+)$ for $i\in\{1,2\}$, must coincide with (ξ_0,η_0) what means (32). □

Corollary 1

With the result of Theorem 1 for the points $A_i = (\xi_i, \eta_i)$ with $i \in \{1,2\}$ calculated in Section 4 therefore remains the following attribution:

If for $i \in \{1,2\}$: $(\xi_0, \eta_0) = ((\xi_i)_+, (\eta_i)_-)$, then $(\xi_i, \eta_i) = ((\xi_i)_-, (\eta_i)_+)$.

If for $i \in \{1,2\}$: $(\xi_0, \eta_0) = ((\xi_i)_-, (\eta_i)_+)$, then $(\xi_i, \eta_i) = ((\xi_i)_+, (\eta_i)_-)$. □

With this information and formulas (23) and (25) the following expressions can be evaluated for $i \in \{1,2\}$:

$$(\xi_0 - d)(\xi_i - d) = \frac{r^2 - d^2}{4r^2}\left(-4y_i^2 + r^2 - d^2\right) \tag{36}$$

$$\eta_0\eta_i = \frac{r^2 - d^2}{4r^2}\left(-4x_i(x_i - d) + r^2 - d^2\right). \tag{37}$$

Both expressions (36) and (37) combine to

$$(\xi_0 - d)(\xi_i - d) + \eta_0\eta_i = -\frac{r^2 - d^2}{2r^2}\left((x_i - d)^2 + y_i^2\right). \tag{38}$$

6. The Third Vertex

We are introducing for $i \in \{1,2\}$ the straight lines

$$(x_i - d)x + y_i y = -\frac{1}{2}\left((x_i - d)^2 + y_i^2\right), \tag{39}$$

which will have a correspondence with the perpendicular bisectors in (5).

Proposition 4

(x_2, y_2) fulfills $(x_1 - d)x + y_1 y = -\frac{1}{2}\left((x_1 - d)^2 + y_1^2\right)$.

(x_1, y_1) fulfills $(x_2 - d)x + y_2 y = -\frac{1}{2}\left((x_2 - d)^2 + y_2^2\right)$.

Proof: We have to show

$$\begin{aligned}(x_1 - d)x_2 + y_1 y_2 &= -\frac{1}{2}\left((x_1 - d)^2 + y_1^2\right)\\ (x_2 - d)x_1 + y_2 y_1 &= -\frac{1}{2}\left((x_2 - d)^2 + y_2^2\right).\end{aligned} \tag{40}$$

Transforming the right hand sides of (40) according to $(x_i - d)^2 + y_i^2 = r^2 - 2dx_i + d^2$ with $i \in \{1,2\}$ both statements of (40) can be equivalently transformed to

$$x_1x_2 + y_1y_2 - d(x_1 + x_2) = -\frac{1}{2}\left(r^2 + d^2\right). \tag{41}$$

It suffices to prove (41). From (11) and (12) we obtain

$$x_1x_2 = \frac{1}{4\left((\xi_0 - d)^2 + \eta_0^2\right)}\left(\left(r^2 - d^2\right)^2 - 4r^2\eta_0^2\right)$$

$$y_1y_2 = \frac{1}{4\left((\xi_0 - d)^2 + \eta_0^2\right)}\left(\left(r^2 - d^2\right)^2 - 4r^2(\xi_0 - d)^2\right)$$

leading to

$$x_1x_2 + y_1y_2 = \frac{1}{4\left(\left(\xi_0 - d\right)^2 + \eta_0^2\right)}\left[2\left(r^2 - d^2\right)^2 - 4r^2\left(\left(\xi_0 - d\right)^2 + \eta_0^2\right)\right]$$
$$= \frac{\left(r^2 - d^2\right)^2}{2\left(\left(\xi_0 - d\right)^2 + \eta_0^2\right)} - r^2.$$

Together with

$$d\left(x_1 + x_2\right) = \frac{2d\left(\xi_0 - d\right)\left(r^2 - d^2\right)}{2\left(\left(\xi_0 - d\right)^2 + \eta_0^2\right)}$$

we get

$$x_1x_2 + y_1y_2 - d\left(x_1 + x_2\right) = \frac{\left(r^2 - d^2\right)^2 - 2d\left(\xi_0 - d\right)\left(r^2 - d^2\right)}{2\left(\left(\xi_0 - d\right)^2 + \eta_0^2\right)} - r^2. \tag{42}$$

The numerator of the fraction in statement (42) can be transformed

$$\left(r^2 - d^2\right)\left(r^2 - d^2 - 2d\left(\xi_0 - d\right)\right) = \left(r^2 - d^2\right)\left(r^2 - 2\xi_0 d + d^2\right)$$
$$= \left(r^2 - d^2\right)\left(\left(\xi_0 - d\right)^2 + \eta_0^2\right).$$

Thus we obtain

$$x_1x_2 + y_1y_2 - d\left(x_1 + x_2\right) = \frac{1}{2}\left(r^2 - d^2\right) - r^2 = -\frac{1}{2}\left(r^2 + d^2\right),$$

which is (41). □

Next the point of intersection $\left(x_T, y_T\right)$ of the straight lines (39) shall be calculated, this means solving the following linear system

$$\begin{aligned}\left(x_1 - d\right)x_T + y_1y_T &= -\frac{1}{2}\left(\left(x_1 - d\right)^2 + y_1^2\right)\\ \left(x_2 - d\right)x_T + y_2y_T &= -\frac{1}{2}\left(\left(x_2 - d\right)^2 + y_2^2\right).\end{aligned} \tag{43}$$

Proposition 5

For the determinant D_T of the linear system (43) holds

$$D_T = \left(x_1 - d\right)y_2 - \left(x_2 - d\right)y_1 = \frac{1}{2}\sqrt{R\left(\xi_0, \eta_0\right)}, \tag{44}$$

which is different from zero for $\left(\xi_0, \eta_0\right)$ with $\xi_0^2 + \eta_0^2 = r^2$.

Proof: Because of (31) we have

$$x_1y_2 - x_2y_1 = \frac{r^2 - d^2}{2\left(\left(\xi_0 - d\right)^2 + \eta_0^2\right)}\sqrt{R\left(\xi_0, \eta_0\right)}$$

and because of (12)

$$-d\left(y_2 - y_1\right) = -\frac{2d\left(\xi_0 - d\right)}{2\left(\left(\xi_0 - d\right)^2 + \eta_0^2\right)}\sqrt{R\left(\xi_0, \eta_0\right)}.$$

This yields for D_T

$$D_T = \frac{r^2 - d^2 - 2d(\xi_0 - d)}{2\left((\xi_0 - d)^2 + \eta_0^2\right)}\sqrt{R(\xi_0, \eta_0)} = \frac{r^2 - 2d\xi_0 + d^2}{2\left((\xi_0 - d)^2 + \eta_0^2\right)}\sqrt{R(\xi_0, \eta_0)}$$

and because of $r^2 - 2d\xi_0 + d^2 = (\xi_0 - d)^2 + \eta_0^2$ formula (44). That $R(\xi_0, \eta_0)$ is positive for (ξ_0, η_0) with $\xi_0^2 + \eta_0^2 = r^2$ was shown in Lemma 1. Therefore also D_T is different from zero for (ξ_0, η_0) with $\xi_0^2 + \eta_0^2 = r^2$. □

Proposition 6

The solutions of the linear system (43) are as follows

$$x_T = d - \frac{(\xi_0 - d)(r^2 - d^2)}{(\xi_0 - d)^2 + \eta_0^2}, \quad y_T = -\frac{\eta_0 (r^2 - d^2)}{(\xi_0 - d)^2 + \eta_0^2}. \tag{45}$$

Proof: Because D_T is different from zero for (ξ_0, η_0) with $\xi_0^2 + \eta_0^2 = r^2$ the linear system (43) has one and only one solution. It is sufficient to show for $i \in \{1, 2\}$

$$(x_i - d)(x_T - d) + y_i y_T = -\frac{1}{2}\left((x_i - d)^2 + y_i^2\right) - d(x_i - d). \tag{46}$$

Substitution of $x_T - d$ and y_T on the left side of (46) yields

$$\begin{aligned}(x_i - d)(x_T - d) + y_i y_T &= -\frac{(x_i - d)(\xi_0 - d)(r^2 - d^2)}{(\xi_0 - d)^2 + \eta_0^2} - \frac{y_i \eta_0 (r^2 - d^2)}{(\xi_0 - d)^2 + \eta_0^2} \\ &= -\frac{r^2 - d^2}{(\xi_0 - d)^2 + \eta_0^2}\left((x_i - d)(\xi_0 - d) + y_i \eta_0\right).\end{aligned}$$

Because of (13) we have

$$\begin{aligned}(x_i - d)(\xi_0 - d) + y_i \eta_0 &= x_i(\xi_0 - d) + y_i \eta_0 - d(\xi_0 - d) \\ &= \frac{1}{2}(r^2 - d^2) - d\xi_0 + d^2 \\ &= \frac{1}{2}(r^2 - 2d\xi_0 + d^2) \\ &= \frac{1}{2}\left((\xi_0 - d)^2 + \eta_0^2\right)\end{aligned}$$

Therefore we get

$$\begin{aligned}(x_i - d)(x_T - d) + y_i y_T &= -\frac{1}{2}(r^2 - d^2) \\ &= -\frac{1}{2}(r^2 - 2dx_i + d^2 + 2dx_i - 2d^2) \\ &= -\frac{1}{2}\left((x_i - d)^2 + y_i^2\right) - d(x_i - d).\end{aligned}$$

which is (46). □

Proposition 7

(x_T, y_T) is located on the circle $x^2 + y^2 = r^2$.

Proof: From (45) we get

$$(x_T - d)^2 + y_T^2 = \frac{(r^2 - d^2)^2\left((\xi_0 - d)^2 + \eta_0^2\right)}{\left((\xi_0 - d)^2 + \eta_0^2\right)^2} = \frac{(r^2 - d^2)^2}{(\xi_0 - d)^2 + \eta_0^2}.$$

Because of $x_T^2 = (x_T - d)^2 + 2dx_T - d^2$ we have

$$x_T^2 + y_T^2 = (x_T - d)^2 + y_T^2 + 2dx_T - d^2 = \frac{(r^2 - d^2)^2}{(\xi_0 - d)^2 + \eta_0^2} + 2dx_T - d^2.$$

Substituting x_T from (45) yields

$$\begin{aligned} x_T^2 + y_T^2 &= \frac{(r^2 - d^2)^2}{(\xi_0 - d)^2 + \eta_0^2} + 2d\left[d - \frac{(\xi_0 - d)(r^2 - d^2)}{(\xi_0 - d)^2 + \eta_0^2}\right] - d^2 \\ &= \frac{(r^2 - d^2)^2 - 2d(\xi_0 - d)(r^2 - d^2)}{(\xi_0 - d)^2 + \eta_0^2} + d^2 \\ &= \frac{(r^2 - d^2)(r^2 - d^2 - 2d(\xi_0 - d))}{(\xi_0 - d)^2 + \eta_0^2} + d^2 \\ &= \frac{(r^2 - d^2)(r^2 - 2d\xi_0 + d^2)}{(\xi_0 - d)^2 + \eta_0^2} + d^2 \\ &= r^2 - d^2 + d^2 = r^2. \end{aligned}$$

□

See **Figure 4**.

Next we want to show that (x_T, y_T) is a point on the perpendicular bisectors of $\overline{DA_i}$ for $i \in \{1,2\}$ given in (5).

Proposition 8

For $i \in \{1,2\}$ holds:

$$(\xi_i - d)x_T + \eta_i y_T = \frac{1}{2}(r^2 - d^2). \tag{47}$$

Proof: Equation (47) can be equivalently rewritten as

$$(\xi_i - d)(x_T - d) + \eta_i y_T = -d(\xi_i - d) + \frac{1}{2}(r^2 - d^2) = \frac{1}{2}(r^2 - 2d\xi_i + d^2). \tag{48}$$

Substituting $x_T - d$ and y_T according to (45) into the left hand side of (48) we get:

$$\begin{aligned} (\xi_i - d)(x_T - d) + \eta_i y_T &= -\frac{(\xi_i - d)(\xi_0 - d)(r^2 - d^2)}{(\xi_0 - d)^2 + \eta_0^2} - \frac{\eta_i\eta_0(r^2 - d^2)}{(\xi_0 - d)^2 + \eta_0^2} \\ &= -\frac{r^2 - d^2}{(\xi_0 - d)^2 + \eta_0^2}\left[(\xi_i - d)(\xi_0 - d) + \eta_i\eta_0\right] \end{aligned} \tag{49}$$

The expression in square brackets of (49) is substituted by (38); this yields

$$= \frac{(r^2 - d^2)^2\left((x_i - d)^2 + y_i^2\right)}{2r^2\left((\xi_0 - d)^2 + \eta_0^2\right)} = \frac{(r^2 - d^2)^2(r^2 - 2dx_i + d^2)}{2r^2(r^2 - 2d\xi_0 + d^2)}. \tag{50}$$

Statement (48) is proven if we can show for the expression on the right hand side of (50)

$$\frac{(r^2 - d^2)^2(r^2 - 2dx_i + d^2)}{2r^2(r^2 - 2d\xi_0 + d^2)} = \frac{1}{2}(r^2 - 2d\xi_i + d^2).$$

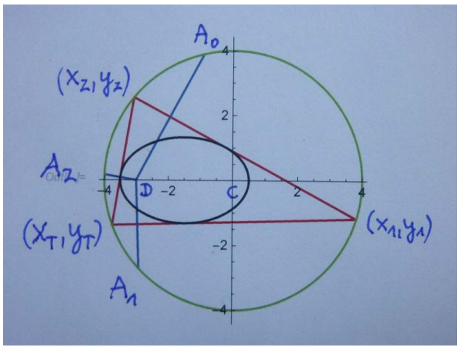

Figure 4. The third vertex (x_T, y_T) located on the circle.

This means

$$\left(r^2-d^2\right)^2\left(r^2-2dx_i+d^2\right)=r^2\left(r^2-2d\xi_0+d^2\right)\left(r^2-2d\xi_i+d^2\right). \quad (51)$$

Statement (51) will be equivalently transformed. The right hand side of (51) can be expanded

$$\begin{aligned} &r^2\left(r^2-d^2-2d\left(\xi_0-d\right)\right)\left(r^2-d^2-2d\left(\xi_i-d\right)\right) \\ &=r^2\left[\left(r^2-d^2\right)^2-2d\left(r^2-d^2\right)\left(\xi_0-d+\xi_i-d\right)+4d^2\left(\xi_0-d\right)\left(\xi_i-d\right)\right]. \end{aligned} \quad (52)$$

Subtracting $r^2\left(r^2-d^2\right)^2$ from the left hand side of (51) and from (52) we get

$$\begin{aligned} &\left(r^2-d^2\right)^2\left(-2dx_i+d^2\right) \\ &=r^2\left[-2d\left(r^2-d^2\right)\left(\xi_0+\xi_i-2d\right)+4d^2\left(\xi_0-d\right)\left(\xi_i-d\right)\right]. \end{aligned} \quad (53)$$

Substituting on the right hand side of (53)

$$\xi_0+\xi_i-2d=\frac{2}{r^2}\left(x_i\left(dx_i+\frac{1}{2}\left(r^2-d^2\right)\right)-r^2d\right)=\frac{2}{r^2}\left(\frac{1}{2}x_i\left(r^2-d^2\right)-dy_i^2\right)$$

according to Corollary 1 and (24) as well as $\left(\xi_0-d\right)\left(\xi_i-d\right)$ according to (36) we get

$$-4d\left(r^2-d^2\right)\left(\frac{1}{2}x_i\left(r^2-d^2\right)-dy_i^2\right)+d^2\left(r^2-d^2\right)\left(-4y_i^2+r^2-d^2\right) \quad (54)$$

The expression $-4d^2\left(r^2-d^2\right)y_i^2$ vanishes in (54). So we have for (53)

$$\begin{aligned} \left(r^2-d^2\right)^2\left(-2dx_i+d^2\right)&=-4d\left(r^2-d^2\right)\frac{1}{2}x_i\left(r^2-d^2\right)+d^2\left(r^2-d^2\right)^2 \\ &=-2d\left(r^2-d^2\right)^2x_i+d^2\left(r^2-d^2\right)^2 \\ &=\left(r^2-d^2\right)^2\left(-2dx_i+d^2\right). \end{aligned} \quad (55)$$

□

Theorem 2

The ellipse created by folding of a disk is enclosed by an arbitrary triangle of tangents, the vertices of which are lying on the circle representing the disk.

Proof: According to Proposition 8 for $i \in \{1,2\}$ the perpendicular bisectors of $\overline{DA_i}$

$$(\xi_i - d)x + \eta_i y = \frac{1}{2}(r^2 - d^2)$$

are passing through (x_T, y_T). Because of their construction in Section 4 they are also going through (x_i, y_i). Thus there is a coincidence with the straight lines (39) in the following way: the straight lines

$$\begin{aligned} (\xi_1 - d)x + \eta_1 y &= \frac{1}{2}(r^2 - d^2) \\ (x_2 - d)x + y_2 y &= -\frac{1}{2}\left((x_2 - d)^2 + y_2^2\right), \end{aligned} \tag{56}$$

having the same points (x_1, y_1) and (x_T, y_T), are thus identical; the straight lines

$$\begin{aligned} (\xi_2 - d)x + \eta_2 y &= \frac{1}{2}(r^2 - d^2) \\ (x_1 - d)x + y_1 y &= -\frac{1}{2}\left((x_1 - d)^2 + y_1^2\right), \end{aligned} \tag{57}$$

having the same points (x_2, y_2) and (x_T, y_T), are thus also identical.

Therefore the ellipse with foci $C = (0,0)$ and $D = (d,0)$ and large semi axis $a = \frac{r}{2}$ is enclosed by a triangle of tangents

$$\begin{aligned} (\xi_0 - d)x + \eta_0 y &= \frac{1}{2}(r^2 - d^2) \\ (\xi_1 - d)x + \eta_1 y &= \frac{1}{2}(r^2 - d^2) \\ (\xi_2 - d)x + \eta_2 y &= \frac{1}{2}(r^2 - d^2) \end{aligned} \tag{58}$$

with vertices (x_1, y_1), (x_2, y_2) and (x_T, y_T) lying on the circle $x^2 + y^2 = r^2$. □

7. Conclusions

It was the intention to show by elementary means of analytic geometry the enclosure of the Gardner ellipse, created by folds of a disk, with an arbitrary triangle of tangents, the vertices of which are lying on the surrounding circle representing the disk.

Remark: The Figures in this paper were constructed with Mathematica using the initial data $r = 4$, $d = -3$, $\xi_0 = -0.9$ and $\eta_0 = +\sqrt{r^2 - \xi_0^2}$.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Gorkin, P. and Shaffer, A. (2021) Making Ellipses by Folding Disks. *Mathematics*

Magazine, **94**, 53-58. https://doi.org/10.1080/0025570X.2021.1849923

[2] Poncelet, J.V. (1865-1866) Traité des propriétés projectives des figures: ouvrage utile à qui s'occupent des applications de la géometrié descriptive et d'opérations géométriques sur le terrain. Vols. 1-2, 2nd Edition, Gauthier-Villars, Paris.

[3] Daepp, U., Gorkin, P. and Mortini, R. (2002) Ellipses and Finite Blaschke Products. *The American Mathematical Monthly*, **109**, 785-795. https://doi.org/10.1080/00029890.2002.11919914

[4] Gardner, M. (1995) New Mathematical Diversions. Revised Edition, Mathematical Association of America, Washington DC.