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Research Article

Classification of Nilpotent Lie Superalgebras of Multiplier-Rank Less than or Equal to 6

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In this paper, we classify all the finite-dimensional nilpotent Lie superalgebras of multiplier-rank less than or equal to 6 over an algebraically closed field of characteristic zero. We also determine the covers of all the nilpotent Lie superalgebras mentioned above.

1. Introduction

The notion of the multiplier $\mathbf{M}(G)$ for a finite group G arose from Schur's work on projective representations of groups. There are fruitful results about this topic. However, we only mention that Green proved that $|\mathbf{M}(G)| \leq p^{(1/2)n(n-1)}$ for a finite p-group G of order p^n (see [1]) as well as that Berkovich and Zhou classified all p-groups with $\mathrm{t}(G) = 0, 1$ and 2, where $\mathrm{t}(G)$ is defined by $|\mathbf{M}(G)| = p^{(1/2)n(n-1)-\mathrm{t}(G)}$ (see [2, 3]).

Batten initiated the theory of multipliers and covers for Lie algebras when she studied the second cohomology groups of Lie algebras with coefficients in a 1-dimensional trivial module (see [4]). For a finite-dimensional Lie algebra L, she proved that $\mathbf{M}(L) \cong \mathbf{H}^2(L, \mathbb{F})$ by a free presentation of L, where $\mathbb{F}$ is the underlying field. Moneyhun gave the maximal dimension of $\mathbf{M}(L)$ (see [5]). Let $\mathrm{t}(L) = (1/2)n(n-1) - \dim \mathbf{M}(L)$, which is a nonnegative integer. Batten et al. classified all nilpotent Lie algebras with $\mathrm{t}(L) = 0, 1, 2$ (see [6]). Hardy and Stitzinger used a different method to get similar results for $\mathrm{t}(L) \leq 8$ (see [7, 8]). For filiform Lie algebras, Bosko classified these up to $\mathrm{t}(L) = 16$ (see [9]).

The notion of the multiplier for Lie algebras may be naturally generalized to the Lie superalgebra case (see [10], for example). Let L be a finite-dimensional Lie superalgebra. A Lie superalgebra pair (K, M) is called a defining pair for L provided that $L \cong K/M$ and $M \subset \mathbf{Z}(K) \cap K^2$. A defining pair (K, M) of L is said to be maximal if K is of maximal super-dimensions. In the case of (K, M) being a maximal defining pair of L, we also call K a cover and M a (Schur) multiplier of L. As in the Lie algebra case, multipliers for Lie superalgebra L are unique up to isomorphism, denoted by $\mathbf{M}(L)$. Moreover, $\mathbf{M}(L) \cong \mathbf{H}^2(L, \mathbb{F})$ (see [11]). By Lemma 2.3 of [10], for a Lie superalgebra L of super-dimension (m, n), we define the *super-multiplier-rank* of L to be

$$\mathrm{smr}(L) = \left(\frac{1}{2}m(m-1) + \frac{1}{2}n(n+1), mn\right) - \mathrm{sdim}\mathbf{M}(L), \tag{1}$$

and the *multiplier-rank* of L to be

$$\mathrm{mr}(L) = |\mathrm{smr}(L)|. \tag{2}$$

In particular, $\mathrm{smr}(L) = (0, 0)$ if and only if L is abelian (see Proposition 3.1 of [10]). Hereafter, we write $\mathrm{sdim}V$ for the super-dimension of a superspace V and $|(a, b)| = a + b$ for a pair (a, b) of nonnegative integers.

In this paper, we classify all nilpotent Lie superalgebras of multiplier-rank ≤6 by discussing whether $\mathbf{Z}(L) \subseteq L^2$. We also construct the covers of all the nilpotent Lie superalgebras of multiplier-rank ≤6.

2. Basics

We assume that all (super)spaces and (super)algebras are over an algebraically closed field $\mathbb{F}$ of characteristic zero. In $\mathbb{Z}\times\mathbb{Z}$, we define a partial order as follows:

$$(m,n)\le(k,l)\Leftrightarrow m\le k, n\le l. \tag{3}$$

We also view $\mathbb{Z}\times\mathbb{Z}$ as the additive group in the usual way.

To classify all the nilpotent Lie superalgebras of multiplier-rank ≤6, we first establish some technical lemmas.

Lemma 1. *Let L be a nilpotent Lie superalgebra of superdimension (m,n). Then, $s\dim L^2\le smr(L)$.*

Proof. By Theorem 3.8 of [11], $\mathbf{M}(L)\cong\mathbf{H}^2(L,\mathbb{F})$. It follows that

$$\mathrm{sdim}\mathbf{Z}^2(L,\mathbb{F})\le\mathrm{sdim}\mathbf{C}^2(L,\mathbb{F})\le\left(\frac{1}{2}m(m-1)+\frac{1}{2}n(n+1),mn\right), \tag{4}$$

where $\mathbf{C}^2(L,\mathbb{F})$ is the superspace consisting of skew-symmetric bilinear functions of L.

Let $\{x_1,\cdots,x_s\mid y_1,\cdots,y_t\}$ be a homogeneous basis of L^2 and extend it to a homogeneous basis of $L:\{x_1,\cdots,x_s,x_{s+1},\cdots,x_m\mid y_1,\cdots,y_t,y_{t+1},\cdots,y_n\}$. Let $x\mapsto x^*$ be the isomorphism from L onto the dual space L^*. For $x^*\in L^*$, let $\hat{x}:L\times L\longrightarrow\mathbb{F}$ be a bilinear mapping by $\hat{x}(y,z)=x^*([y,z])$ for all $y,z\in L$. Then, $\hat{x}\in\mathbf{B}^2(L,\mathbb{F})$. If $\hat{x}=0$, then $x^*|_{[L,L]}=0$. Then, $x=0$ for all $x\in L^2$. Hence, $x\mapsto\hat{x}$ is an injection from L^2 into $\mathbf{B}^2(L,\mathbb{F})$. Consequently, $\mathrm{sdim}\mathbf{M}(L)\le((1/2)m(m-1)+(1/2)n(n+1),mn)-\mathrm{sdim}L^2$. By the definition of $\mathrm{smr}(L)$, we have $\mathrm{sdim}L^2\le\mathrm{smr}(L)$.□ □

Let Π be the parity functor of superspaces. Note that

$$\mathrm{sdim}V+\mathrm{sdim}\Pi V=(\dim V,\dim V). \tag{5}$$

Lemma 2. *Let L be a nilpotent Lie superalgebra. Suppose K is a 1-dimensional central ideal of L and $K\subseteq L^2$. The following statements hold.*

(1) If $sdimK=(1,0)$, then $smr(L/K)+sdimL^2\le smr(L)$

(2) If $sdimK=(0,1)$, then $smr(L/K)+sdim\Pi L^2+(0,1)\le smr(L)$

Proof. By Lemma 4.9 of [11], we have an exact sequence:

$$0\longrightarrow\mathbf{Hom}\left(\frac{L}{K},\mathbb{F}\right)\xrightarrow{I}\mathbf{Hom}(L,\mathbb{F})\xrightarrow{R}\mathbf{Hom}(K,\mathbb{F})\xrightarrow{T}\mathbf{M}\left(\frac{L}{K}\right)\xrightarrow{I}\mathbf{M}(L)\xrightarrow{\delta}\frac{L}{L^2}\otimes K. \tag{6}$$

Since $K\subseteq L^2$, we have $R(f)(k)=f\circ i(k)\in f(L^2)=0$ for all $k\in K$ and $f\in\mathbf{Hom}(L,\mathbb{F})$, where $i:K\longrightarrow L$ is a Lie superalgebra monomorphism. Then, $R=0$ and T is an injection. Hence, $\mathrm{sdim}(\mathrm{im}T)=\mathrm{sdim}K=(1,0)$ or $(0,1)$. Furthermore, $\mathrm{sdim}\mathbf{M}(L)=\mathrm{sdim}(\mathrm{im}I)+\mathrm{sdim}(\mathrm{im}\delta)$ and $\mathrm{sdim}\mathbf{M}(L/K)=\mathrm{sdim}(\mathrm{im}I)+\mathrm{sdim}(\mathrm{im}T)$.

(1) If $\mathrm{sdim}(\mathrm{im}T)=\mathrm{sdim}K=(1,0)$, we have

$$\begin{aligned}\mathrm{sdim}\mathbf{M}(L)+(1,0)&=\mathrm{sdim}\mathbf{M}(L)+\mathrm{sdim}(\mathrm{im}T)\\&=\mathrm{sdim}(\mathrm{im}I)+\mathrm{sdim}(\mathrm{im}\delta)+\mathrm{sdim}(\mathrm{im}T)\\&=\mathrm{sdim}\mathbf{M}\left(\frac{L}{K}\right)+\mathrm{sdim}(\mathrm{im}\delta)\\&\le\mathrm{sdim}\left(L/L^2\otimes K\right)+\mathrm{sdim}\mathbf{M}\left(\frac{L}{K}\right)\\&=\mathrm{sdim}\left(\frac{L}{L^2}\right)+\mathrm{sdim}\mathbf{M}\left(\frac{L}{K}\right).\end{aligned} \tag{7}$$

By the definition of $\mathrm{smr}(L)$, we have $\mathrm{smr}(L/K)+\mathrm{sdim}\,L^2\le\mathrm{smr}(L)$.

(2) If $\mathrm{sdim}(\mathrm{im}T)=\mathrm{sdim}K=(0,1)$, we have

$$\begin{aligned}\mathrm{sdim}\mathbf{M}(L)+(0,1)&\le\mathrm{sdim}\left(\frac{L}{L^2}\otimes K\right)+\mathrm{sdim}\mathbf{M}\left(\frac{L}{K}\right)\\&=\mathrm{sdim}\Pi\left(\frac{L}{L^2}\right)+\mathrm{sdim}\mathbf{M}\left(\frac{L}{K}\right).\end{aligned} \tag{8}$$

Hence, $\mathrm{smr}(L/K)+\mathrm{sdim}\Pi L^2+(0,1)\le\mathrm{smr}(L)$.□ □

As in the Lie algebra case (Theorem 1 of [7]), using free presentations of Lie superalgebras, one may prove the following lemma.

Lemma 3. *Let A and B be finite-dimensional Lie superalgebras. Then,*

$$sdim\mathbf{M}(A\oplus B)=sdim\mathbf{M}(A)+sdim\mathbf{M}(B)+sdim\left(\frac{A}{A^2}\otimes\frac{B}{B^2}\right). \tag{9}$$

Lemma 4. *Let L be a finite-dimensional nilpotent Lie superalgebra. Suppose that $\mathbf{Z}(L)UL^2$. Then, there exists an ideal A of L with $\dim A=1$ such that $L=M\oplus A$, where M is an ideal of L.*

(1) If $sdimA=(1,0)$, then $smr(L)=smr(M)+sdimM^2$

(2) If $sdimA=(0,1)$, then $smr(L)=smr(M)+sdim\Pi M^2+(1,0)$

Proof. Suppose that A is a one-dimensional subsuperspace of L such that $A\subseteq\mathbf{Z}(L)$ but AUL^2. Let M be a complement to A in L such that $L^2\subseteq M$. Then, $L=M\oplus A$ and $\mathbf{M}(A)=(0,0)$.

If $\operatorname{sdim} A = (1, 0)$, then $\operatorname{sdim}(M/M^2 \otimes A) = \operatorname{sdim} M - \operatorname{sdim} M^2$. By Lemma 3, we have

$$\operatorname{sdim}\mathbf{M}(L) = \operatorname{sdim}\mathbf{M}(A \oplus M) = \operatorname{sdim}\mathbf{M}(A) + \operatorname{sdim}\mathbf{M}(M) + \operatorname{sdim}\left(\frac{A}{A^2} \otimes \frac{M}{M^2}\right). \tag{10}$$

Then, $\operatorname{smr}(L) = \operatorname{smr}(M) + \operatorname{sdim} M^2$.

If $\operatorname{sdim} A = (0, 1)$, then $\operatorname{sdim}(M/M^2 \otimes A) = \operatorname{sdim} \Pi M - \operatorname{sdim} \Pi M^2$. Now $((1/2)m(m-1) + (1/2)n(n+1), mn) - \operatorname{smr}(L) = ((1/2)m(m-1) + (1/2)(n-1)n, m(n-1)) - \operatorname{smr}(M) + (n-1, m) - \operatorname{sdim} \Pi M^2$. Therefore, $\operatorname{smr}(L) = \operatorname{smr}(M) + \operatorname{sdim} \Pi M^2 + (1, 0)$.□ □

For convenience, we write $\mathrm{Ab}(m, n)$ for the abelian Lie superalgebra of super-dimension (m, n), $\mathrm{H}(p, q)$ for the $(2p+1, q)$-super-dimensional Heisenberg Lie superalgebra of even center, and $\mathrm{H}(k)$ for the $(k, k+1)$-super-dimensional Heisenberg Lie superalgebra of odd center (see [12]). In Proposition 4.4 of [10] and section 4 of [13], the authors characterize the multipliers of $\mathrm{H}(p, q)$ and $\mathrm{H}(k)$:

$$\operatorname{sdim}\mathbf{M}(\mathrm{H}(p,q)) = \begin{cases} \left(2p^2 - p + \frac{1}{2}q^2 + \frac{1}{2}q - 1, 2pq\right), & p + q \geq 2, \\ (0, 0), & p = 0, q = 1, \\ (2, 0), & p = 1, q = 0, \end{cases}$$
$$\operatorname{sdim}\mathbf{M}(\mathrm{H}(k)) = \begin{cases} (k^2, k^2 - 1), & k \geq 2, \\ (1, 1), & k = 1. \end{cases} \tag{11}$$

Hence,

$$\operatorname{smr}\mathrm{H}(p, q) = (2p + 1, q) = \operatorname{sdim}\mathrm{H}(p, q), p + q \geq 2, \tag{12}$$

$$\operatorname{smr}\mathrm{H}(k) = (k + 1, k + 1), k \geq 2, \operatorname{smr}\mathrm{H}(1) = (1, 2). \tag{13}$$

Similarly, let us give a multiplier and cover of $\mathrm{H}(p, q) \oplus \mathrm{Ab}(s, t)$.

Case 1. Suppose that $(p, q) = (1, 0)$. Let $\widehat{\mathrm{H}(1,0) \oplus \mathrm{Ab}}(\mathrm{s}, \mathrm{t})$ be a Lie superalgebra with a homogenous basis

$$\{x, y, z, a_k, \gamma_k, \delta, \eta_k, \theta, \lambda_{c,n}, \sigma_{d,o} \mid b_m, \varepsilon_m, \vartheta_m, \mu_{k,m}\}, \tag{14}$$

and multiplications

$$\begin{aligned} &[x, y] = z, && [x, a_k] = \gamma_k, && [x, z] = \delta, \\ &[x, b_m] = \varepsilon_m, && [y, a_k] = \eta_k, && [y, z] = \theta, \\ &[y, b_m] = \vartheta_m, && [a_c, a_n] = \lambda_{c,n}, && [a_k, b_m] = \mu_{k,m}, \\ &[b_d, b_o] = \sigma_{d,o}, \end{aligned} \tag{15}$$

where $1 \leq c, k, n \leq s, 1 \leq d, m, o \leq t$, and $c < n, d \leq o$. Let $\mathrm{M}\widehat{\mathrm{H}(1,0) \oplus \mathrm{Ab}}(s, t)$ be a subsuperspace spanned by $\{\gamma_k, \delta, \eta_k, \theta, \lambda_{c,n}, \sigma_{d,o} \mid \varepsilon_m, \vartheta_m, \mu_{k,m}\}$. Then,

$$0 \longrightarrow \mathrm{M}\widehat{\mathrm{H}(1,0) \oplus \mathrm{Ab}}(s, t) \longrightarrow \widehat{H(1,0) \oplus \mathrm{Ab}}(s, t) \longrightarrow \mathrm{H}(1,0) \oplus \mathrm{Ab}(s, t) \longrightarrow 0, \tag{16}$$

is a maximal stem extension of $\mathrm{H}(1, 0) \oplus \mathrm{Ab}(s, t)$. In particular, $\mathrm{M}\widehat{\mathrm{H}(1,0) \oplus \mathrm{Ab}}(s, t)$ is a multiplier and $\widehat{\mathrm{H}(1,0) \oplus \mathrm{Ab}}(s, t)$ is a cover of $\mathrm{H}(1, 0) \oplus \mathrm{Ab}(s, t)$.

Case 2. Suppose that $(p, q) \neq (1, 0)$. Let $\widehat{\mathrm{H}(p,q) \oplus \mathrm{Ab}}(\mathrm{s}, \mathrm{t})$ be a Lie superalgebra with even basis

$$\left\{x_i, y_i, z, a_k, \alpha_{e,j}, \alpha_i, \beta_{f,h}, \gamma_{i,k}, \zeta_{e,j}, \eta_{i,k}, \lambda_{c,n}, \sigma_{d,o}, \varsigma_{m,l}, \omega_{u,r}, \rho_l\right\}, \tag{17}$$

and odd basis

$$\{w_l, b_m, \varepsilon_{i,m}, \varepsilon_{i,l}, \vartheta_{i,m}, \iota_{i,l}, \mu_{k,m}, \nu_{k,l}\}, \tag{18}$$

and multiplications

$$\begin{aligned} &2[x_e, x_j] = \alpha_{e,j}, && [x_i, y_i] = z + \alpha_i, \\ &[x_f, y_h] = \beta_{f,h}, && [x_i, a_k] = \gamma_{i,k}, \\ &[x_i, b_m] = \varepsilon_{i,m}, && [x_i, w_l] = \varepsilon_{i,l}, \\ &[y_e, y_j] = \zeta_{e,j}, && [y_i, a_k] = \eta_{i,k}, \\ &[y_i, b_m] = \vartheta_{i,m}, && [y_i, w_l] = \iota_{i,l}, \\ &[a_c, a_n] = \lambda_{c,n}, && [a_k, b_m] = \mu_{k,m}, \\ &[a_k, w_l] = \nu_{k,l}, && [b_d, b_o] = \sigma_{d,o}, \\ &[b_m, w_l] = \varsigma_{m,l}, && [w_u, w_r] = \omega_{u,r}, \\ &[w_l, w_l] = z + \rho_l, \end{aligned} \tag{19}$$

where $1 \leq e, f, h, i, j \leq p, 1 \leq c, k, n \leq s, 1 \leq d, m, o \leq t, 1 \leq l, r, u \leq q$, and $e < j, f \neq h, c < n, d \leq o$, and $u < r$. Let $\mathrm{M}\widehat{\mathrm{H}(p,q) \oplus \mathrm{Ab}}(s, t)$ be a subsuperspace spanned by

$$\left\{\alpha_{e,j}, \alpha_i, \beta_{f,h}, \gamma_{i,k}, \zeta_{e,j}, \eta_{i,k}, \lambda_{c,n}, \sigma_{d,o}, \varsigma_{m,l}, \omega_{u,r}, \rho_l \mid \varepsilon_{i,m}, \varepsilon_{i,l}, \vartheta_{i,m}, \iota_{i,l}, \mu_{k,m}, \nu_{k,l}\right\}. \tag{20}$$

Then,

$$0 \longrightarrow \mathrm{M}\widehat{\mathrm{H}(p,q) \oplus \mathrm{Ab}}(s, t) \longrightarrow \widehat{\mathrm{H}(p,q) \oplus \mathrm{Ab}}(s, t) \longrightarrow \mathrm{H}(p, q) \oplus \mathrm{Ab}(s, t) \longrightarrow 0, \tag{21}$$

is a maximal stem extension of $\mathrm{H}(p, q) \oplus \mathrm{Ab}(s, t)$. In particular, $\mathrm{M}\widehat{\mathrm{H}(p,q) \oplus \mathrm{Ab}}(s, t)$ is a multiplier and $\widehat{\mathrm{H}(p,q) \oplus \mathrm{Ab}}(s, t)$ is a cover of $\mathrm{H}(p, q) \oplus \mathrm{Ab}(s, t)$.

Summarizing, we have the following.

Lemma 5. *Let p, q, s, t be positive integers. Then,*

$$0 \longrightarrow M\widehat{H(p,q) \oplus Ab}(s,t) \longrightarrow \widehat{H(p,q) \oplus Ab}(s,t) \longrightarrow H(p,q) \oplus Ab(s,t) \longrightarrow 0, \tag{22}$$

is a maximal stem extension of $H(p,q) \oplus Ab(s,t)$. In particular, $\widehat{H(p,q) \oplus Ab}(s,t)$ is the cover of $H(p,q) \oplus Ab(s,t)$ and the super-dimension of $\mathbf{M}(H(p,q) \oplus Ab(s,t))$ as follows:

$$\begin{gathered} 12s2 + 12t2 + 32s + 12t + 2, st + 2t, p, q = 1, 0, \\ \left(\tilde{p} + \tilde{q} + \tilde{s} + \tilde{t} - 1, 2pq + 2pt + st + sq\right), (p,q) \neq (1,0), \end{gathered} \tag{23}$$

where $\tilde{p} = 2p^2 - p + 2ps$, $\tilde{q} = (1/2)q^2 + (1/2)q$, $\tilde{s} = (1/2)s^2 - (1/2)s$, $\tilde{t} = (1/2)t^2 + (1/2)t + tq$.

3. Multiplier-Rank 3 Nilpotent Lie Superalgebras

In this section, we will determine all the nilpotent Lie superalgebras of $\mathrm{mr}(L) = 3$. The following theorem is analogous to the one in the Lie algebra case [7], yet it contains more information in our super-case.

Theorem 6. *Let L be a finite-dimensional, nonabelian, and nilpotent Lie superalgebra of $mr(L) = 3$. Then,*

(1) $smr(L) \neq (0,3)$

(2) $smr(L) = (1,2)$ if and only if $L \cong H(0,2)$

(3) $smr(L) = (3,0)$ if and only if $L \cong H(1,0) \oplus Ab(2,0)$

(4) $smr(L) = (2,1)$ if and only if L is isomorphic to one of the following Lie superalgebras:

(a) $H(1,0) \oplus Ab(1,1)$

(b) $H(0,1) \oplus Ab(1,0)$

(c) $H(1)$

Proof. Let us characterize L by discussing whether $\mathbf{Z}(L) \subseteq L^2$.

(1) Suppose that $\mathbf{Z}(L) \subseteq L^2$. Let K be an ideal contained in $\mathbf{Z}(L)$ with dim $K = 1$. Take $H = L/K$

(a) If $\mathrm{sdim}K = (1,0)$, then one may check that $\mathrm{smr}(L) = (1,2)$ and $\mathrm{sdim}L^2 = (1,0)$ by Lemma 2. In this case, by (12), we have

$$L \cong \mathrm{H}(0,2). \tag{24}$$

(b) If $\mathrm{sdim}K = (0,1)$, then Lemma 2 yields that $\mathrm{smr}(L) = (2,1)$. Hence, $\mathrm{smr}(H) + \mathrm{sdim}\Pi L^2 \leq (2,0)$. Since L is not abelian, we have $\mathrm{sdim}\Pi L^2 \neq (0,0)$. If $\mathrm{sdim}\,\Pi L^2 = (2,0)$, then $\mathrm{sdim}H^2 = (0,1)$ and $\mathrm{smr}(H) = (0,0)$, an impossibility. Therefore, $\mathrm{sdim}\Pi L^2 \neq (2,0)$. It remains to consider the case $\mathrm{sdim}\Pi L^2 = (1,0)$. In this case, we have $L \cong \mathrm{H}(k)$ for some k. By (13), we have $L \cong \mathrm{H}(1)$

(2) Suppose that $\mathbf{Z}(L)\mathrm{U}L^2$. By Lemma 4, we have $L = M \oplus A$, where A and M are ideals of L with dim $A = 1$

(a) If $\mathrm{sdim}A = (0,1)$, then one may check that there are no algebras satisfying Lemma 2 (2) for $\mathrm{smr}(L) = (3,0), (2,1), (1,2)$, and $(0,3)$

(b) If $\mathrm{sdim}A = (1,0)$, one has to discuss the cases $\mathrm{smr}(L) = (3,0), (0,3), (2,1)$, and $(1,2)$

(i) Assume that $\mathrm{smr}(L) = (3,0)$. We obtain that $\mathrm{smr}(M) = (2,0)$ and $\mathrm{sdim}M^2 = (1,0)$ by Lemmas 1 and 4 (1). It follows that $M \cong \mathrm{H}(1,0) \oplus \mathrm{Ab}(1,0)$ by Theorem 5.8 of [10]. Hence,

$$L \cong \mathrm{H}(1,0) \oplus \mathrm{Ab}(2,0). \tag{25}$$

By Lemma 5, (3) is proven.

(ii) Assume that $\mathrm{smr}(L) = (0,3)$. We have $\mathrm{smr}(M) = (0,2)$ and $\mathrm{sdim}M^2 = (0,1)$ by Lemmas 1 and 4 (1). There are no such algebras for $\mathrm{smr}(M) = (0,2)$, by Proposition 5.22 of [10]. Therefore, (1) is proven

(iii) Assume that $\mathrm{smr}(L) = (2,1)$. By Lemmas 1 and 4 (1), we have $\mathrm{smr}(M) = (1,1)$, which yields $\mathrm{sdim}M^2 = (1,0)$. It follows that $M \cong \mathrm{H}(1,0) \oplus \mathrm{Ab}(0,1)$ or $\mathrm{H}(0,1)$. Hence,

$$L \cong \mathrm{H}(1,0) \oplus \mathrm{Ab}(1,1) \text{ or } \mathrm{H}(0,1) \oplus \mathrm{Ab}(1,0). \tag{26}$$

Then, (4) holds by Lemma 5.

(iv) Assume that $\mathrm{smr}(L) = (1,2)$. We have either $\mathrm{smr}(M) = (1,2)$ or $\mathrm{smr}(M) = (1,1)$ by Lemmas 1 and 4 (1). If $\mathrm{smr}(M) = (1,2)$, we have $\mathrm{sdim}M^2 = (0,0)$, an impossibility. If $\mathrm{smr}(M) = (1,1)$, then we have $\mathrm{sdim}M^2 = (0,1)$. This is impossible, because $\mathrm{smr}(M) = (1,1)$ yields $\mathrm{sdim}M^2 = (1,0)$ by Theorem 5.8 of [10]. Then the, proof is complete

□ □

4. Multiplier-Rank 4 Nilpotent Lie Superalgebras

In this section, we will determine all the nilpotent Lie superalgebras of $\mathrm{mr}(L) = 4$. We recall that $L(3,4,1,4)$ is a Lie algebra with basis $\{x, y, z, r\}$ and nonzero multiplication $[x, y] = z$, $[x, z] = r$ and that $L(4, 5, 2, 4)$ is a Lie algebra with basis $\{x, y, z, r, c\}$ and nonzero multiplication $[x, y] = z$, $[x, c] = r$ (see [7]).

Theorem 7. *Let L be a finite-dimensional, nonabelian, and nilpotent Lie superalgebra of $mr(L) = 4$. Then,*

(1) *$smr(L) \neq (0, 4)$*

(2) *$smr(L) = (1, 3)$ if and only if $L \cong H(0, 3)$*

(3) *$smr(L) = (2, 2)$ if and only if L is isomorphic to one of the following Lie superalgebras:*

  (a) *$H(1, 0) \oplus Ab(0, 2)$*

  (b) *$H(0, 1) \oplus Ab(0, 1)$*

  (c) *$H(1) \oplus Ab(1, 0)$*

(4) *$smr(L) = (4, 0)$ if and only if L is isomorphic to one of the following Lie superalgebras:*

  (a) *$H(1, 0) \oplus Ab(3, 0)$*

  (b) *$L(3, 4, 1, 4)$*

  (c) *$L(4, 5, 2, 4)$*

(5) *$smr(L) = (3, 1)$ if and only if L is isomorphic to one of the following Lie superalgebras:*

  (a) *$H(1, 0) \oplus Ab(2, 1)$*

  (b) *$H(0, 1) \oplus Ab(2, 0)$*

  (c) *$H(1, 1)$*

Proof. Let us characterize L by discussing whether $\mathbf{Z}(L) \subseteq L^2$.

(1) Suppose that $\mathbf{Z}(L) \subseteq L^2$. We may assume that $K \subseteq \mathbf{Z}(L)$, where $\dim K = 1$ and $H \cong L/K$

  (a) If $\mathrm{sdim}(K) = (0, 1)$, one may check that $\mathrm{smr}(L) = (2, 2)$ by Lemma 2. Now assume that $\mathrm{smr}(L) = (2, 2)$. Then, $\mathrm{sdim}\Pi L^2 + \mathrm{smr}(H) \leq (2, 1)$. Since $\mathrm{sdim}H^2 \leq \mathrm{smr}(H)$, hence $\mathrm{sdim}L^2 = (1, 1)$ or $(0, 1)$. In the first case $\mathrm{sdim}L^2 = (1, 1)$, we obtain that $\mathrm{sdim}H^2 = (1, 0)$ and $\mathrm{smr}(H) = (1, 0)$. Hence, $H \cong \mathrm{H}(1, 0)$. Then, L can be described generally by basis $\{x, y, z \mid k\}$ with multiplication given by $[x, y] = z$. To compute the multiplier start with

$$\begin{gathered}[x, y] = z + s_1, [x, z] = s_2, [x, k] = s_3, \\ [y, z] = s_4, [y, k] = s_5, [z, k] = s_6, \\ [k, k] = s_7,\end{gathered} \tag{27}$$

where $s_1, s_2, \cdots, s_7$ generate the multiplier. A change of variables allows that $s_1 = 0$. Use of the Jacobi identity on all possible triples shows that $s_6 = 0$ and $\mathbf{M}(L) = \langle s_2, s_4, s_7 \mid s_3, s_5 \rangle$. Hence, $\mathrm{smr}(L) = (1, 1)$, contradicting the assumption $\mathrm{smr}(L) = (2, 2)$. In the other case $\mathrm{sdim}L^2 = (0, 1)$, we obtain that $L^2 = \mathbf{Z}(L)$. Then, $L \cong \mathrm{H}(k)$ for some k. There are no such algebras by (13) for $\mathrm{smr}(L) = (2, 2)$

  (b) If $\mathrm{sdim}K = (1, 0)$, then Lemmas 1 and 2 yield $\mathrm{smr}(L) = (4, 0)$,$(3, 1)$ and $(1, 3)$

    (i) Assume that $\mathrm{smr}(L) = (4, 0)$. As in Lie algebra case (Theorem 2 of [7]), one can determine that

$$L \cong L(3, 4, 1, 4) \text{ or } L(4, 5, 2, 4). \tag{28}$$

    (ii) Assume that $\mathrm{smr}(L) = (3, 1)$ and $(1, 3)$. The possible case for $\mathrm{sdim}L^2$ is $(1, 0)$. It follows that $L^2 = \mathbf{Z}(L)$ and $L \cong \mathrm{H}(p, q)$ for some p, q. By (12), we have

$$L \cong \mathrm{H}(1, 1), \mathrm{H}(0, 3), \tag{29}$$

respectively

(2) Suppose that $\mathbf{Z}(L)\mathrm{U}L^2$. By Lemma 4, we have $L = A \oplus M$, where A and M are ideals of L with $\dim A = 1$

  (a) If $\mathrm{sdim}A = (0, 1)$, then Lemmas 1 and 4 (2) yield that $\mathrm{smr}(L) = (2, 2)$. Therefore, $\mathrm{smr}(M) = (1, 1)$ and $\mathrm{sdim}M^2 = (1, 0)$. Hence,

$$L \cong \mathrm{H}(1, 0) \oplus \mathrm{Ab}(0, 2) \text{ or } \mathrm{H}(0, 1) \oplus \mathrm{Ab}(0, 1). \tag{30}$$

  (b) If $\mathrm{sdim}A = (1, 0)$, then Lemmas 1 and 4 (1) yield that $\mathrm{smr}(L) = (4, 0), (2, 2)$ or $(3, 1)$. As a result of the above, we have proven (1) and (2). To prove (3), (4), and (5), one has to discuss the cases $\mathrm{smr}(L) = (4, 0), (2, 2)$, and $(3, 1)$

    (i) Assume that $\mathrm{smr}(L) = (4, 0)$. It is easily checked that there are no such algebras except for $\mathrm{sdim}M^2 = (1, 0)$ and $\mathrm{smr}(M) = (3, 0)$. By Theorem 6, we have $M \cong \mathrm{H}(1, 0) \oplus \mathrm{Ab}(2, 0)$ and

$$L \cong \mathrm{H}(1, 0) \oplus \mathrm{Ab}(3, 0). \tag{31}$$

By Lemma 5, (4) is proven.

(ii) Assume that $\mathrm{smr}(L) = (2, 2)$. We may check that $\mathrm{smr}(M) = (2, 1)$ and $\mathrm{sdim} M^2 = (0, 1)$. Then, $M \cong \mathrm{H}(1)$ by Theorem 6. Hence,

$$L \cong \mathrm{H}(1) \oplus \mathrm{Ab}(1, 0). \tag{32}$$

Then, (3) holds by Lemma 5

(iii) Assume that $\mathrm{smr}(L) = (3, 1)$. We have $\mathrm{smr}(M) = (2, 1)$ and $\mathrm{sdim} M^2 = (1, 0)$. By Theorem 6, we have $M \cong \mathrm{H}(1, 0) \oplus \mathrm{Ab}(1, 1)$ or $\mathrm{H}(0, 1) \oplus \mathrm{Ab}(1, 0)$. Then,

$$L \cong \mathrm{H}(1, 0) \oplus \mathrm{Ab}(2, 1) \text{ or } \mathrm{H}(0, 1) \oplus \mathrm{Ab}(2, 0). \tag{33}$$

The proof is complete.□ □

5. Multiplier-Rank 5 Nilpotent Lie Superalgebras

In this section, we will determine all the nilpotent Lie superalgebras of $\mathrm{mr}(L) = 5$. For convenience, let L_1 denote a Lie superalgebra with a homogeneous basis $\{x, k \mid z, y\}$ and nonzero multiplication $[x, y] = z$, $[z, y] = k$.

Theorem 8. *Let L be a finite-dimensional, nonabelian, and nilpotent Lie superalgebra of $mr(L) = 5$. Then,*

(1) $smr(L) \neq (0, 5)$ and $(2, 3)$

(2) $smr(L) = (1, 4)$ if and only if $L \cong H(0, 4)$

(3) $smr(L) = (5, 0)$ if and only if L is isomorphic to one of the following Lie superalgebras:

(a) $H(1, 0) \oplus Ab(4, 0)$

(b) $H(2, 0)$

(4) $smr(L) = (4, 1)$ if and only if L is isomorphic to one of the following Lie superalgebras:

(a) $H(1, 0) \oplus Ab(3, 1)$

(b) $H(0, 1) \oplus Ab(3, 0)$

(c) $H(1, 1) \oplus Ab(1, 0)$

(5) $smr(L) = (3, 2)$ if and only if L is isomorphic to one of the following Lie superalgebras:

(a) $H(1, 0) \oplus Ab(1, 2)$

(b) $H(0, 1) \oplus Ab(1, 1)$

(c) $H(1, 2)$

(d) L_1

Proof. Let us characterize L by discussing whether $\mathbf{Z}(L) \subseteq L^2$.

(1) Suppose that $\mathbf{Z}(L) \subseteq L^2$. We may assume that $K \subseteq \mathbf{Z}(L)$, where $\dim K = 1$ and $H \cong L/K$

(a) If $\mathrm{sdim} K = (1, 0)$, then Lemma 2 readily yields $\mathrm{smr}(L) = (5, 0)$, $(3, 2)$ or $(1, 4)$

(i) Assume that $\mathrm{smr}(L) = (5, 0)$. Then, from the proof of Theorem 3 of [3], we have

$$L \cong \mathrm{H}(2, 0). \tag{34}$$

(ii) Assume that $\mathrm{smr}(L) = (3, 2)$. This readily yields $\mathrm{sdim} L^2 = (1, 0)$, $(2, 0)$, or $(1, 1)$. In the first case $\mathrm{sdim} L^2 = (1, 0)$, then $L^2 = \mathbf{Z}(L)$ and $L \cong \mathrm{H}(p, q)$ for some p, q. By (12), we have

$$L \cong \mathrm{H}(1, 2). \tag{35}$$

Consider the second case $\mathrm{sdim} L^2 = (2, 0)$. If $\mathrm{smr}(H) = (0, 2)$ and $(0, 1)$, then there are no such algebras by the previous work. If $\mathrm{smr}(H) = (1, 1)$, then $H \cong \mathrm{H}(1, 0) \oplus \mathrm{Ab}(0, 1)$ or $\mathrm{H}(0, 1)$. If $H \cong \mathrm{H}(1, 0) \oplus \mathrm{Ab}(0, 1)$, then a computation shows that $\mathrm{smr}(L) = (5, 2)$, contradicting the assumption $\mathrm{smr}(L) = (3, 2)$. If $H \cong \mathrm{H}(0, 1)$, we have $\mathrm{smr}(L) = (2, 1)$ by a direct computation, also contradicting the assumption $\mathrm{smr}(L) = (3, 2)$. If $\mathrm{smr}(H) = (1, 0)$, then $H \cong \mathrm{H}(1, 0)$. By the proof of Theorem 1 of [7], we have $L \cong L(3, 4, 1, 4)$ and $\mathrm{smr}(L) = (4, 0)$. This contradicts the assumption on $\mathrm{smr}(L)$. If $\mathrm{smr}(H) = (1, 2)$, then $H \cong \mathrm{H}(0, 2)$, which contradicts the assumption $\mathrm{smr}(L) = (3, 2)$. In the third case $\mathrm{sdim} L^2 = (1, 1)$, we have $\mathrm{sdim} H^2 = (0, 1)$. By the previous work, we have $\mathrm{smr}(H) = (2, 1)$ and $H \cong \mathrm{H}(1)$. Then, L has a homogeneous basis $\{x, k \mid z, y\}$ with multiplication given by $[x, y] = z$, $[z, z] = a_1 k$, $[y, y] = a_2 k$, $[z, y] = a_3 k$, where $a_1, a_2, a_3 \in \mathbb{F}$, $k \in \mathbf{Z}(L)$. The Jacobi identity shows that $a_1 = 0$. Since $\mathrm{sdim} L^2 = (1, 1)$, we have that either a_2 or a_3 is not zero. Without loss of generality, assume that $a_3 \neq 0$. Replacing z by $-(1/2a_3)z$ and y by $-2y + (a_2/a_3)z$ and relabeling, we get a simple multiplication table:

$$[x, y] = z, [z, y] = k. \tag{36}$$

To compute the multiplier, we start with

$$\begin{aligned} &[x, k] = s_1, && [x, z] = s_2, && [x, y] = z + s_3, \\ &[k, z] = s_4, && [k, y] = s_5, && [z, z] = s_6, \\ &[z, y] = k + s_7, && [y, y] = s_8, \end{aligned} \tag{37}$$

where $s_1, s_2, \cdots, s_8$ generate the multiplier. By relabeling, we get $s_3 = s_7 = 0$. Using the Jacobi identity gives $s_1 = s_4 = s_6 = 0$. Hence, the multiplier has a homogeneous basis $\{s_8 \mid s_2, s_5\}$, and then, $\mathrm{sdim} \mathbf{M}(L) = (1, 2)$ and $\mathrm{smr}(L) = (3, 2)$. This superalgebra satisfies the requirements. As mentioned above, we have

$$L \cong L_1. \quad (38)$$

(iii) Assume that $\mathrm{smr}(L) = (1, 4)$. If $\mathrm{sdim}L^2 \geq (1, 0)$, then $\mathrm{smr}(H) \leq (0, 3)$. There are no such algebras by the previous work. Then, $\mathrm{sdim}L^2 = (1, 0)$. It follows from Proposition 4.11 of [10] that

$$L \cong \mathrm{H}(0, 4). \quad (39)$$

(b) If $\mathrm{sdim}(K) = (0, 1)$, then Lemma 2 (2) yields that $\mathrm{smr}(L) = (3, 2)$ or $(2, 3)$

(i) Assume that $\mathrm{smr}(L) = (3, 2)$. Then, $\mathrm{sdim}\Pi L^2 + \mathrm{smr}(H) \leq (3, 1)$. It follows that $\mathrm{sdim}L^2 = (1, 1)$ and $\mathrm{smr}(H) \leq (2, 0)$. If $\mathrm{smr}(H) = (2, 0)$, then $H \cong \mathrm{H}(1, 0) \oplus \mathrm{Ab}(1, 0)$. Computing the multiplier as before yields $\mathrm{sdim}\mathbf{M}(L) = (5, 3)$ and $\mathrm{smr}(L) = (2, 1)$, contradicting the assumption $\mathrm{smr}(L) = (3, 2)$. If $\mathrm{smr}(H) = (1, 0)$, then $H \cong \mathrm{H}(1, 0)$. It is also a contradiction by the previous work

(ii) Assume that $\mathrm{smr}(L) = (2, 3)$. We have $\mathrm{sdim}\Pi L^2 + \mathrm{smr}(H) \leq (2, 2)$. It follows that $\mathrm{sdimL}^2 = (1, 1)$ and $\mathrm{smr}(H) = (1, 1)$. Then, $H \cong \mathrm{H}(1, 0) \oplus \mathrm{Ab}(0, 1)$ or $\mathrm{H}(0, 1)$. If $H \cong \mathrm{H}(1, 0) \oplus \mathrm{Ab}(0, 1)$, then $\mathrm{sdim}\mathbf{M}(L) = (4, 2)$ and $\mathrm{smr}(L) = (2, 4)$ by computing $\mathbf{M}(L)$ as above, contradicting the assumption $\mathrm{smr}(L) = (2, 3)$. If $H \cong \mathrm{H}(0, 1)$, then L has a homogeneous basis $\{z \mid x, k\}$ with multiplication given by $[x, x] = z$. Computing the multiplier as before yields $\mathrm{sdim}\mathbf{M}(L) = (1, 1)$ and $\mathrm{smr}(L) = (2, 1)$, also contradicting the assumption $\mathrm{smr}(L) = (2, 3)$

(2) Suppose that $\mathbf{Z}(L) \mathrm{U} L^2$. By Lemma 4, we have $L = A \oplus M$, where A and M are ideals of L with $\dim A = 1$

(a) If $\mathrm{sdim}A = (0, 1)$, then Lemma 4(2) yields that $\mathrm{smr}(L) = (3, 2)$ and $\mathrm{smr}(M) = (2, 1)$. Then by Theorem 6, we have

$$L \cong \mathrm{H}(1, 0) \oplus \mathrm{Ab}(1, 2) \text{ or } \mathrm{H}(0, 1) \oplus \mathrm{Ab}(1, 1). \quad (40)$$

(b) If $\mathrm{sdim}A = (1, 0)$, then Lemmas 1 and 4 yield that $\mathrm{smr}(L) = (5, 0), (4, 1)$ or $(3, 2)$. As a result of the above, we have proven (1) and (2). To prove (3), (4), and (5), one has to discuss the cases $\mathrm{smr}(L) = (5, 0), (4, 1)$, and $(3, 2)$

(i) Assume that $\mathrm{smr}(L) = (5, 0)$. Then, $\mathrm{sdim}M^2 \leq (2, 0)$. If $\mathrm{sdim}M^2 = (2, 0)$, then $\mathrm{smr}(M) = (3, 0)$. By Theorem 6, we have $M \cong \mathrm{H}(1, 0) \oplus \mathrm{Ab}(2, 0)$. This yields $\mathrm{sdim}M^2 = (1, 0)$, contradicting the assumption $\mathrm{sdim}M^2 = (2, 0)$. If $\mathrm{sdim}M^2 = (0, 0)$, then M is abelian and $\mathrm{smr}(M) = (5, 0)$, an impossibility. If $\mathrm{sdim}M^2 = (1, 0)$. Then, $\mathrm{smr}(M) = (4, 0)$. By Theorem 7 (4), we have $M \cong \mathrm{H}(1, 0) \oplus \mathrm{Ab}(3, 0)$, $L(3, 4, 1, 4)$ or $L(4, 5, 2, 4)$. Since $\mathrm{sdim}M^2 = (1, 0)$, we have $M \cong \mathrm{H}(1, 0) \oplus \mathrm{Ab}(3, 0)$. Then,

$$L \cong \mathrm{H}(1, 0) \oplus \mathrm{Ab}(4, 0). \quad (41)$$

By Lemma 5, (3) holds.

(ii) Assume that $\mathrm{smr}(L) = (4, 1)$. Then, $(0, 0) < \mathrm{sdim}M^2 \leq (2, 0)$. If $\mathrm{sdim}M^2 = (2, 0)$, computing the multiplier as before yields $\mathrm{smr}(L) = (2, 1)$. This contradicts the assumption on $\mathrm{smr}(L)$. Therefore, $\mathrm{sdim}M^2 = (1, 0)$ and $\mathrm{smr}(M) = (3, 1)$. Then, M is one of the Lie superalgebras listed in Theorem 7 (5). Therefore, L is isomorphic to one of the following Lie superalgebras:

$$\mathrm{H}(1, 0) \oplus \mathrm{Ab}(3, 1), \mathrm{H}(0, 1) \oplus \mathrm{Ab}(3, 0), \mathrm{H}(1, 1) \oplus \mathrm{Ab}(1, 0). \quad (42)$$

Then, (4) holds by Lemma 5.

(iii) Assume that $\mathrm{smr}(L) = (3, 2)$. Then, $(0, 0) < \mathrm{sdim}M^2 \leq (1, 1)$. If $\mathrm{sdim}M^2 = (0, 1)$, then $\mathrm{smr}(M) = (3, 1)$. This is impossible, because the superdimension of the derived superalgebra of M when $\mathrm{smr}(M) = (3, 1)$ is not $(0, 1)$. Similarly, if $\mathrm{sdim}M^2 = (1, 1)$, then $\mathrm{smr}(M) = (2, 1)$, a contradiction. Now consider $\mathrm{sdim}M^2 = (1, 0)$; we have $\mathrm{smr}(M) = (2, 2)$. By Theorem 7 (3), we have

$$L \cong \mathrm{H}(1, 0) \oplus \mathrm{Ab}(1, 2) \text{ or } \mathrm{H}(0, 1) \oplus \mathrm{Ab}(1, 1). \quad (43)$$

The proof is complete.□ □

6. Multiplier-Rank 6 Nilpotent Lie Superalgebras

In this section, we will determine all the nilpotent Lie superalgebras of $\mathrm{mr}(L) = 6$. Let us recall that $L(4, 5, 1, 6)$ is a Lie algebra with basis $\{x, y, z, r, c\}$ and nonzero multiplication $[x, y] = z$, $[x, z] = r$, $[y, c] = r$. For convenience, let L_2 denote a Lie superalgebra with a homogeneous basis $\{a, z \mid x, k\}$ and nonzero multiplication $[x, x] = z$, $[a, x] = k$.

Theorem 9. *Let L be a finite-dimensional, nonabelian, and nilpotent Lie superalgebra of $mr(L)=6$. Then,*

(1) *$smr(L) \neq (0,6)$*

(2) *$smr(L)=(1,5)$ if and only if $L \cong H(0,5)$*

(3) *$smr(L)=(2,4)$ if and only if $L \cong H(0,4) \oplus Ab(1,0)$*

(4) *$smr(L)=(6,0)$ if and only if L is isomorphic to one of the following Lie superalgebras:*

  (a) *$H(1,0) \oplus Ab(5,0)$*

  (b) *$H(2,0) \oplus Ab(1,0)$*

  (c) *$L(3,4,1,4) \oplus Ab(1,0)$*

  (d) *$L(4,5,2,4) \oplus Ab(1,0)$*

  (e) *$L(4,5,1,6)$*

(5) *$smr(L)=(5,1)$ if and only if L is isomorphic to one of the following Lie superalgebras:*

  (a) *$H(1,0) \oplus Ab(4,1)$*

  (b) *$H(0,1) \oplus Ab(4,0)$*

  (c) *$H(1,1) \oplus Ab(2,0)$*

  (d) *$H(2,1)$*

(6) *$smr(L)=(4,2)$ if and only if L is isomorphic to one of the following Lie superalgebras:*

  (a) *$H(1,0) \oplus Ab(2,2)$*

  (b) *$H(0,1) \oplus Ab(2,1)$*

  (c) *$H(1,2) \oplus Ab(1,0)$*

  (d) *$H(1,1) \oplus Ab(0,1)$*

(7) *$smr(L)=(3,3)$ if and only if L is isomorphic to one of the following Lie superalgebras:*

  (a) *$H(1,0) \oplus Ab(0,3)$*

  (b) *$H(0,1) \oplus Ab(0,2)$*

  (c) *$H(1,3)$*

  (d) *L_2*

  (e) *$H(2)$*

Proof. Let us characterize L by discussing whether $Z(L) \subseteq L^2$.

(1) Suppose that $\mathbf{Z}(L) \subseteq L^2$. We may assume that $K \subseteq \mathbf{Z}(L)$, where $\dim K = 1$ and $H \cong L/K$

  (a) If $\mathrm{sdim} K = (1,0)$, then Lemma 2 readily yields $\mathrm{smr}(L) = (6,0), (5,1), (3,3)$, or $(1,5)$. In the first case $\mathrm{smr}(L) = (6,0)$, by Theorem 4 of [7], one can determine

$$L \cong L(4,5,1,6). \tag{44}$$

In the remaining cases $\mathrm{smr}(L) = (5,1), (3,3)$, and $(1,5)$, we obtain that the possible super-dimension of L^2 is $(1,0)$. It follows that $L^2 = \mathbf{Z}(L)$ and $L \cong \mathrm{H}(p,q)$ for some p, q. By (12), we have

$$L \cong \mathrm{H}(2,1), \mathrm{H}(1,3), \mathrm{H}(0,5), \tag{45}$$

respectively.

  (b) If $\mathrm{sdim}(K) = (0,1)$, then Lemma 2 yields that $\mathrm{smr}(L) = (3,3)$ or $(2,4)$

    (i) Assume that $\mathrm{smr}(L) = (3,3)$. Then, $\mathrm{sdim}\Pi L^2 + \mathrm{smr}(H) \leq (3,2)$. It follows that the only possible super-dimension of L^2 is $(1,1)$. Hence, we have $\mathrm{sdim} H^2 = (1,0)$ and $\mathrm{smr}(H) \leq (2,1)$. There are no such algebras for $\mathrm{smr}(H) < (2,1)$ by the previous work. If $\mathrm{smr}(H) = (2,1)$, then $H \cong \mathrm{H}(1,0) \oplus \mathrm{Ab}(1,1)$ or $\mathrm{H}(0,1) \oplus \mathrm{Ab}(1,0)$. In the first case $H \cong \mathrm{H}(1,0) \oplus \mathrm{Ab}(1,1)$, then L has a homogeneous basis $\{x, y, z, a \mid b, k\}$ with multiplication $[x,y] = z$, $[x,b] = a_2 k$, $[y,b] = a_3 k$, $[z,b] = a_4 k$, $[a,b] = a_5 k$. The Jacobi identity shows that $a_4 = 0$. Since $a \notin L^2$ and $\mathbf{Z}(L) \subseteq L^2$, we have $a_5 \neq 0$. Take $(1/a_5)a$ for a. If $a_2 = a_3 = 0$, then we obtain a simple multiplication $[x,y] = z$, $[a,b] = k$. Computing the multiplier as before yields $\mathrm{sdim}\mathbf{M}(L) = (5,3)$ and $\mathrm{smr}(L) = (4,5)$, contradicting the assumption $\mathrm{smr}(L) = (3,3)$. If a_2 or a_3 is not zero, then without loss of generality, assume that $a_2 \neq 0$. Replacing x by $(1/a_2)x$ and y by $-a_3 x + a_2 y$ and relabeling, we obtain a simple multiplication $[x,y] = z$, $[x,b] = k$, $[a,b] = k$. Computing the multiplier as before yields $\mathrm{sdim}\mathbf{M}(L) = (6,1)$ and $\mathrm{smr}(L) = (3,7)$, also contradicting the assumption $\mathrm{smr}(L) = (3,3)$. In the other case $H \cong \mathrm{H}(0,1) \oplus \mathrm{Ab}(1,0)$, one may obtain that L can be described by a homogeneous basis $\{a, z \mid x, k\}$ with multiplication $[x,x] = z$, $[a,x] = a_2 k$, $[z,x] = a_3 k$. The Jacobi identity shows that $a_3 = 0$. Take $(1/a_2)a$ for a, relabel and get $[x,x] = z$, $[a,x] = k$. Then, $\mathrm{sdim}\mathbf{M}(L) = (1,1)$ and $\mathrm{smr}(L) = (3,3)$. This algebra satisfies the requirements. As mentioned above, we have

$$L \cong L_2. \tag{46}$$

    (ii) Assume that $\mathrm{smr}(L) = (2,4)$. Then, $\mathrm{sdim}\Pi L^2 + \mathrm{smr}(H) \leq (2,3)$ and $\mathrm{sdim} L^2 = (0,1)$ by a direct computation. Hence, $L^2 = \mathbf{Z}(L)$. It follows that $L \cong \mathrm{H}(k)$ for some k. By (12), we have

$$L \cong \mathrm{H}(2). \tag{47}$$

(2) Suppose that $\mathbf{Z}(L)\mathrm{U}L^2$. By Lemma 4, we have $L = A \oplus M$, where A and M are ideals of L with dim $A =$ 1

(a) If sdim$A = (0, 1)$, then Lemmas 1 and 4 (2) yield that smr$(L) = (5, 1), (4, 2)$, or $(3, 3)$. Then, sdim $\Pi M^2 = (0, 1)$ by the previous work

(i) Assume that smr$(L) = (5, 1)$. Then, smr$(M) = (4, 0)$ and Theorem 7 reveals three candidates, only one of which satisfies sdim$M^2 = (1, 0)$. Therefore, $L \cong \mathrm{H}(1, 0) \oplus \mathrm{Ab}(3, 1)$. It yields smr$(L) = (4, 1)$ and contradicts the assumption that smr$(L) = (5, 1)$

(ii) Assume that smr$(L) = (4, 2)$. Then, smr$(M) = (3, 1)$ and $M \cong \mathrm{H}(1, 0) \oplus \mathrm{Ab}(2, 1), \mathrm{H}(0, 1) \oplus \mathrm{Ab}(2, 0)$, or $\mathrm{H}(1, 1)$. Therefore, L is isomorphic to one of the following Lie superalgebras:

$$\mathrm{H}(1, 0) \oplus \mathrm{Ab}(2, 2), \mathrm{H}(0, 1) \oplus \mathrm{Ab}(2, 1), \mathrm{H}(1, 1) \oplus \mathrm{Ab}(0, 1). \tag{48}$$

(iii) Assume that smr$(L) = (3, 3)$. Then, smr$(M) = (2, 2)$ and $M \cong \mathrm{H}(1, 0) \oplus \mathrm{Ab}(0, 2)$ or $\mathrm{H}(0, 1) \oplus \mathrm{Ab}(0, 1)$. Hence,

$$L \cong \mathrm{H}(1, 0) \oplus \mathrm{Ab}(0, 3) \text{ or } \mathrm{H}(0, 1) \oplus \mathrm{Ab}(0, 2). \tag{49}$$

(b) If sdim$A = (1, 0)$, then Lemmas 1 and 4 (1) yield that the possible super-multiplier-rank of L is $(2, 4), (6, 0), (5, 1)$, or $(4, 2)$. By the above work, we have proven (1), (2), and (7). To prove (3), (4), (5), and (6), one has to discuss the cases smr$(L) = (2, 4), (6, 0), (5, 1)$, and $(4, 2)$

(i) Assume that smr$(L) = (2, 4)$. One may obtain that smr$(M) = (1, 4)$ and sdim$M^2 = (1, 0)$. By Theorem 8, we have $M \cong \mathrm{H}(0, 4)$ and

$$L \cong \mathrm{H}(0, 4) \oplus \mathrm{Ab}(1, 0). \tag{50}$$

Then, (3) holds by Lemma 5.

(ii) Assume that smr$(L) = (6, 0)$. By Theorem 4 of [7], we have L is isomorphic to one of the following Lie superalgebras:

$$\begin{gathered} L(3, 4, 1, 4) \oplus \mathrm{Ab}(1, 0), \mathrm{H}(1, 0) \oplus \mathrm{Ab}(5, 0), \\ L(4, 5, 2, 4) \oplus \mathrm{Ab}(1, 0), \mathrm{H}(2, 0) \oplus \mathrm{Ab}(1, 0). \end{gathered} \tag{51}$$

By Lemma 5, (4) is proven.

(iii) Assume that smr$(L) = (5, 1)$. Since sdim$M^2 \leq$ smr(M), it follows that smr$(M) \geq (3, 1)$. If smr$(M) = (5, 1)$, then M is abelian and sdim$M^2 = (0, 0)$, an impossibility. Therefore, smr$(M) \neq (5, 1)$. If smr$(M) = (4, 1)$, then $M \cong \mathrm{H}(1, 0) \oplus \mathrm{Ab}(3, 1), \mathrm{H}(0, 1) \oplus \mathrm{Ab}(3, 0)$, or $\mathrm{H}(1, 1) \oplus \mathrm{Ab}(1, 0)$ by Theorem 8. Therefore L is isomorphic to one of the following Lie superalgebras:

$$\mathrm{H}(1, 0) \oplus \mathrm{Ab}(4, 1), \mathrm{H}(0, 1) \oplus \mathrm{Ab}(4, 0), \mathrm{H}(1, 1) \oplus \mathrm{Ab}(2, 0). \tag{52}$$

If smr$(M) = (3, 1)$, then $M \cong \mathrm{H}(1, 0) \oplus \mathrm{Ab}(2, 1), \mathrm{H}(0, 1) \oplus \mathrm{Ab}(2, 0)$, or $\mathrm{H}(1, 1)$ and sdim$M^2 = (2, 0)$. These are impossible, because the super-dimension of the derived superalgebras of $\mathrm{H}(1, 0) \oplus \mathrm{Ab}(2, 1), \mathrm{H}(0, 1) \oplus \mathrm{Ab}(2, 0)$, and $\mathrm{H}(1, 1)$ are $(1, 0)$. Hence, (5) holds by Lemma 5.

(iv) Assume that smr$(L) = (4, 2)$. The only possibility of super-multiplier-rank of M is $(3, 2)$. Then, Theorem 7 reveals four candidates, only three of which satisfy sdim$M^2 = (1, 0)$. Therefore, L is isomorphic to one of the following Lie superalgebras:

$$\mathrm{H}(1, 0) \oplus \mathrm{Ab}(2, 2), \mathrm{H}(0, 1) \oplus \mathrm{Ab}(2, 1), \mathrm{H}(1, 2) \oplus \mathrm{Ab}(1, 0). \tag{53}$$

The proof is complete.□ □

7. Covers

By Theorem 3.3 of [13], we obtain that any two covers of a finite-dimensional Lie superalgebra are isomorphic as ordinary Lie superalgebras. In this section, we will describe the covers of all the nilpotent Lie superalgebras of multiplier-rank ≤6.

First, we compute the cover for $L(3, 4, 1, 4) \oplus \mathrm{Ab}(1, 0)$. Let $\{x_1, x_2, x_3, x_4, u \mid |x_i| = |u| = \bar{0}, i = 1, 2, 3, 4\}$ be a homogeneous basis of $L(3, 4, 1, 4) \oplus \mathrm{Ab}(1, 0)$, where $x_1, x_2, x_3, x_4 \in L(3, 4, 1, 4)$ and $u \in \mathrm{Ab}(1, 0)$. The multiplication is given by $[x_1, x_2] = x_3, [x_1, x_3] = x_4$, the other brackets of basis elements vanishing. Suppose that

$$0 \longrightarrow M \longrightarrow K \longrightarrow L(3, 4, 1, 4) \oplus \mathrm{Ab}(1, 0) \longrightarrow 0, \tag{54}$$

is a stem extension of $L(3, 4, 1, 4) \oplus \mathrm{Ab}(1, 0)$. Then, M

Table 1: The classification and covers.

smr(L)	L	Cover
$(0,0)$	Any abelian Lie superalgebra $\mathrm{Ab}(s,t)$, $s,t\in\mathbb{N}$	$\widehat{\mathrm{Ab}}(s,t)$
$(1,0)$	$\mathrm{H}(1,0)$	$\widehat{\mathrm{H}}(1,0)$
$(2,0)$	$\mathrm{H}(1,0)\oplus\mathrm{Ab}(1,0)$	$\widehat{\mathrm{H}(1,0)\oplus\mathrm{Ab}(1,0)}$
$(1,1)$	$\mathrm{H}(1,0)\oplus\mathrm{Ab}(0,1)$	$\widehat{\mathrm{H}(1,0)\oplus\mathrm{Ab}(1,0)}$
	$\mathrm{H}(0,1)$	$\widehat{\mathrm{H}}(0,1)$
$(1,2)$	$\mathrm{H}(0,2)$	$\widehat{\mathrm{H}}(0,2)$
$(3,0)$	$\mathrm{H}(1,0)\oplus\mathrm{Ab}(2,0)$	$\widehat{\mathrm{H}(1,0)\oplus\mathrm{Ab}(2,0)}$
$(2,1)$	$\mathrm{H}(1,0)\oplus\mathrm{Ab}(1,1)$	$\widehat{\mathrm{H}(1,0)\oplus\mathrm{Ab}(1,1)}$
	$\mathrm{H}(0,1)\oplus\mathrm{Ab}(1,0)$	$\widehat{\mathrm{H}(0,1)\oplus\mathrm{Ab}(1,0)}$
	$\mathrm{H}(1)$	$\widehat{\mathrm{H}}(1)$
$(1,3)$	$\mathrm{H}(0,3)$	$\widehat{\mathrm{H}}(0,3)$
$(2,2)$	$\mathrm{H}(1,0)\oplus\mathrm{Ab}(0,2)$	$\widehat{\mathrm{H}(1,0)\oplus\mathrm{Ab}(0,2)}$
	$\mathrm{H}(0,1)\oplus\mathrm{Ab}(0,1)$	$\widehat{\mathrm{H}(0,1)\oplus\mathrm{Ab}(0,1)}$
$(4,0)$	$\mathrm{H}(1,0)\oplus\mathrm{Ab}(3,0)$	$\widehat{\mathrm{H}(1,0)\oplus\mathrm{Ab}(3,0)}$
	$L(3,4,1,4)$	$\widehat{L(3,4,1,4)}$
	$L(4,5,2,4)$	$\widehat{L(4,5,2,4)}$
$(3,1)$	$\mathrm{H}(1,0)\oplus\mathrm{Ab}(2,1)$	$\widehat{\mathrm{H}(1,0)\oplus\mathrm{Ab}(2,1)}$
	$\mathrm{H}(0,1)\oplus\mathrm{Ab}(2,0)$	$\widehat{\mathrm{H}(0,1)\oplus\mathrm{Ab}(2,0)}$
	$\mathrm{H}(1,1)$	$\widehat{\mathrm{H}}(1,1)$
$(1,4)$	$\mathrm{H}(0,4)$	$\widehat{\mathrm{H}}(0,4)$
$(5,0)$	$\mathrm{H}(1,0)\oplus\mathrm{Ab}(4,0)$	$\widehat{\mathrm{H}(1,0)\oplus\mathrm{Ab}(4,0)}$
	$\mathrm{H}(2,0)$	$\widehat{\mathrm{H}}(2,0)$
$(4,1)$	$\mathrm{H}(1,0)\oplus\mathrm{Ab}(3,1)$	$\widehat{\mathrm{H}(1,0)\oplus\mathrm{Ab}(3,1)}$
	$\mathrm{H}(0,1)\oplus\mathrm{Ab}(3,0)$	$\widehat{\mathrm{H}(0,1)\oplus\mathrm{Ab}(3,0)}$
	$\mathrm{H}(1,1)\oplus\mathrm{Ab}(1,0)$	$\widehat{\mathrm{H}(1,1)\oplus\mathrm{Ab}(1,0)}$
$(3,2)$	$\mathrm{H}(1,0)\oplus\mathrm{Ab}(1,2)$	$\widehat{\mathrm{H}(1,0)\oplus\mathrm{Ab}(1,2)}$
	$\mathrm{H}(0,1)\oplus\mathrm{Ab}(1,1)$	$\widehat{\mathrm{H}(0,1)\oplus\mathrm{Ab}(1,1)}$
	$\mathrm{H}(1,2)$	$\widehat{\mathrm{H}}(1,2)$
	L_1	$\widehat{L_1}$
$(2,4)$	$\mathrm{H}(0,4)\oplus\mathrm{Ab}(1,0)$	$\widehat{\mathrm{H}(0,4)\oplus\mathrm{Ab}(1,0)}$
$(6,0)$	$\mathrm{H}(1,0)\oplus\mathrm{Ab}(5,0)$	$\widehat{\mathrm{H}(1,0)\oplus\mathrm{Ab}(5,0)}$
	$\mathrm{H}(2,0)\oplus\mathrm{Ab}(1,0)$	$\widehat{\mathrm{H}(2,0)\oplus\mathrm{Ab}(1,0)}$
	$L(3,4,1,4)\oplus\mathrm{Ab}(1,0)$	$\widehat{L(3,4,1,4)\oplus\mathrm{Ab}(1,0)}$
	$L(4,5,2,4)\oplus\mathrm{Ab}(1,0)$	$\widehat{L(4,5,2,4)\oplus\mathrm{Ab}(1,0)}$
	$L(4,5,1,6)$	$\widehat{L(4,5,1,6)}$
$(5,1)$	$\mathrm{H}(1,0)\oplus\mathrm{Ab}(4,1)$	$\widehat{\mathrm{H}(1,0)\oplus\mathrm{Ab}(5,0)}$
	$\mathrm{H}(0,1)\oplus\mathrm{Ab}(4,0)$	

Table 1: Continued.

smr(L)	L	Cover
		$\mathrm{H}(0,1)\widehat{\oplus \mathrm{Ab}}(4,0)$
	$\mathrm{H}(1,1)\oplus \mathrm{Ab}(2,0)$	$\mathrm{H}(1,1)\widehat{\oplus \mathrm{Ab}}(2,0)$
	$\mathrm{H}(2,1)$	$\widehat{\mathrm{H}}(2,1)$
	$\mathrm{H}(1,0)\oplus \mathrm{Ab}(2,2)$	$\mathrm{H}(1,0)\widehat{\oplus \mathrm{Ab}}(2,2)$
	$\mathrm{H}(0,1)\oplus \mathrm{Ab}(2,1)$	$\mathrm{H}(0,1)\widehat{\oplus \mathrm{Ab}}(2,1)$
(4,2)	$\mathrm{H}(1,2)\oplus \mathrm{Ab}(1,0)$	$\mathrm{H}(1,2)\widehat{\oplus \mathrm{Ab}}(1,0)$
	$\mathrm{H}(1,1)\oplus \mathrm{Ab}(0,1)$	$\mathrm{H}(1,1)\widehat{\oplus \mathrm{Ab}}(0,1)$
	$\mathrm{H}(1,0)\oplus \mathrm{Ab}(0,3)$	$\mathrm{H}(1,0)\widehat{\oplus \mathrm{Ab}}(0,3)$
	$\mathrm{H}(0,1)\oplus \mathrm{Ab}(0,2)$	$\mathrm{H}(0,1)\widehat{\oplus \mathrm{Ab}}(0,2)$
(3,3)	$\mathrm{H}(1,3)$	$\widehat{\mathrm{H}}(1,3)$
	L_2	$\widehat{L_2}$
	$\mathrm{H}(2)$	$\widehat{\mathrm{H}}(2)$
(1, 5)	$\mathrm{H}(0,5)$	$\widehat{\mathrm{H}}(0,5)$

$\subset K^2 \cap \mathbf{Z}(K)$ and $K/M \cong L(3,4,1,4)\oplus \mathrm{Ab}(1,0)$. Then, K/M has a homogeneous basis $\{x_1+M, x_2+M, x_3+M, x_4+M, u+M\}$. We may assume that

$$\begin{aligned} &[x_1,x_2]=x_3+a_1, \quad [x_1,x_3]=x_4+a_2, \quad [x_1,x_4]=a_3,\\ &[x_2,x_3]=a_4, \quad [x_2,x_4]=a_5, \quad [x_3,x_4]=a_6,\\ &[x_1,u]=a_7, \quad [x_2,u]=a_8, \quad [x_3,u]=a_9,\\ &[x_4,u]=a_{10}, \end{aligned} \tag{55}$$

where $a_1,\cdots,a_{10}\in M_1$. Without loss of generality, we may suppose that $a_1=a_2=0$. By the Jacobi identity of Lie superalgebras, we have $a_5=a_6=a_9=a_{10}=0$. Then, M is spanned by a_3,a_4,a_7,a_8 and K is spanned by $x_1,x_2,x_3,x_4,u,a_3,a_4,a_7,a_8$. Hence, $\mathrm{sdim}M\le(4,0)$. Now let $L(3,4,1,\widehat{4)\oplus}\mathrm{Ab}(1,0)$ be a Lie superalgebra with a homogenous basis $\{x,y,z,r,u,a,b,c,d\}$ and multiplication $[x,y]=z,[x,z]=r,[x,r]=a,[y,z]=b,[x,u]=c,[y,u]=d$. Let $ML(3,4,1,\widehat{4)\oplus}\mathrm{Ab}(1,0)$ be a subsuperspace of $L(3,4,1,\widehat{4)\oplus}\mathrm{Ab}(1,0)$ spanned by a,b,c,d. Then,

$$\begin{aligned} &ML(3,4,1,\widehat{4)\oplus}\mathrm{Ab}(1,0)\subseteq \mathbf{Z}\left(L(3,4,1,\widehat{4)\oplus}\mathrm{Ab}(1,0)\right)\cap\left(L(3,4,1,\widehat{4)\oplus}\mathrm{Ab}(1,0)\right)^2,\\ &\frac{L(3,4,1,4)\oplus\mathrm{Ab}(1,0)}{ML(3,4,1,4)\oplus\mathrm{Ab}(1,0)}\cong L(3,4,1,4)\oplus\mathrm{Ab}(1,0). \end{aligned} \tag{56}$$

Since $\mathrm{sdim}(ML(3,4,1,\widehat{4)\oplus}\mathrm{Ab}(1,0))=(4,0)$, we obtain that $ML(3,4,1,\widehat{4)\oplus}\mathrm{Ab}(1,0)$ is a multiplier and $L(3,4,1,\widehat{4)\oplus}\mathrm{Ab}(1,0)$ is a cover of $L(3,4,1,4)\oplus\mathrm{Ab}(1,0)$.

Then, we compute the cover for $L(3,4,1,4)$. Let $L(3,\widehat{4,1},4)$ be a Lie superalgebra with a homogenous basis $\{x,y,z,r,a,b\}$ and multiplication $[x,y]=z,[x,z]=r,[x,r]=a,[y,z]=b$. Since, $L(3,4,1,4)$ is a subalgebra of $L(3,4,1,4)\oplus\mathrm{Ab}(1,0)$, one may check that $L(3,\widehat{4,1},4)$ is a cover of $L(3,4,1,4)$.

Now we compute the cover for $L(4,5,2,4)\oplus\mathrm{Ab}(1,0)$. Let $\{x,y,z,c,r,v\mid |x|=|y|=|z|=|c|=|r|=|v|=\bar{0}\}$ be a homogeneous basis of $L(4,5,2,4)\oplus\mathrm{Ab}(1,0)$, where $x,y,z,c,r\in L(4,5,2,4)$ and $v\in\mathrm{Ab}(1,0)$. The multiplication is given by $[x,y]=z,[x,c]=r$, the other brackets of basis elements vanishing. Similarly, one may check that the Lie superalgebra with a homogeneous basis $\{x,y,z,c,r,v,a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8\}$ and multiplication

$$\begin{aligned} &[x,y]=z, \quad [x,z]=a_1, \quad [x,c]=r\\ &[x,r]=a_2, \quad [y,z]=a_3, \quad [y,c]=a_4,\\ &[c,r]=a_5, \quad [x,v]=a_6, \quad [y,v]=a_7,\\ &[c,v]=a_8 \end{aligned} \tag{57}$$

is a cover of $L(4,5,2,4)\oplus\mathrm{Ab}(1,0)$, which is denoted by $L(4,5,2,\widehat{4)\oplus}\mathrm{Ab}(1,0)$.

Then, we compute the cover for $L(4,5,2,4)$. Since $L(4,5,2,4)$ is a subalgebra of $L(4,5,2,4)\oplus\mathrm{Ab}(1,0)$, one may check that the Lie superalgebra with a homogeneous basis $\{x,y,z,c,r,a_1,a_2,a_3,a_4,a_5\}$ and multiplication $[x,y]=z,[x,z]=a_1,[x,c]=r,[x,r]=a_2,[y,z]=a_3,[y,c]=a_4,[c,r]=a_5$ is a cover of $L(4,5,2,4)$, which is denoted by $L(4,\widehat{5,2},4)$.

When $\mathrm{smr}(L)=(6,0)$, we also obtain a cover $L(4,\widehat{5,1},6)$ for $L(4,5,1,6)$, which has a homogenous basis

$\{x, y, z, r, c, a_1, a_2, a_3, a_4\}$ and multiplication

$$[x, y] = z, \quad [x, z] = r, \quad [x, r] = a_1, \\ [x, c] = a_2, \quad [y, z] = a_3, \quad [y, c] = r + a_4. \tag{58}$$

By the previous work, we may obtain a cover $\widehat{L_1}$ for L_1, which has a homogenous basis $\{x, k, s_8 \mid z, y, s_2, s_5\}$ and multiplication $[x, y] = z, [z, y] = k, [x, z] = s_2, [k, y] = s_5, [y, y] = s_8$. When $\mathrm{smr}(L) = (3, 3)$, we may obtain a cover $\widehat{L_2}$ for L_2, which has a homogenous basis $\{a, z, b \mid c, x, k\}$ and multiplication $[a, x] = k, [a, k] = c, [x, x] = z, [x, k] = b$.

8. Main Result

Theorem 10. *The classification and covers of all the finite-dimensional nilpotent Lie superalgebras L of multiplier-rank ≤6 are listed as follows:*

where L_1 is a (2, 2)-superdimensional Lie superalgebra with a homogeneous basis $\{x, k \mid z, y\}$ and nonzero multiplication $[x, y] = z, [z, y] = k$,L_2 is a (2, 2)-superdimensional Lie superalgebra with a homogeneous basis $\{a, z \mid x, k\}$ and nonzero multiplication $[x, x] = z, [a, x] = k$.

Proof. By Theorem 5.8 of [10], we obtain all the finite-dimensional nilpotent Lie superalgebras L of $\mathrm{mr}(L) \leq 2$. In Theorems 6, 7, 8, and 9, all the nilpotent Lie superalgebras of $\mathrm{mr}(L) = 3, 4, 5, 6$ are determined. We only need to describe the covers of all Lie superalgebras of $\mathrm{mr}(L) \leq 6$. By Theorem 4.1 of [13], Lemma 5, and Section 7, we may obtain the covers of all the nilpotent Lie superalgebras of $\mathrm{mr}(L) \leq 6$. In summary, we obtain Table 1.□ □

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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