Journal of High Energy Physics, Gravitation and Cosmology, 2024, 10, 930-957
https://www.scirp.org/journal/jhepgc
ISSN Online: 2380-4335
ISSN Print: 2380-4327

Legendre-Jacobi's Elliptic Integrals Shed Light on the Luminosity Distance in Cosmology

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How to cite this paper: Trinchera, A. (2024) Legendre-Jacobi's Elliptic Integrals Shed Light on the Luminosity Distance in Cosmology. *Journal of High Energy Physics, Gravitation and Cosmology*, **10**, 930-957.
https://doi.org/10.4236/jhepgc.2024.103057

Received: March 17, 2024
Accepted: June 30, 2024
Published: July 3, 2024

Abstract

This article concerns the integral related to the transverse comoving distance and, in turn, to the luminosity distance both in the standard non-flat and flat cosmology. The purpose is to determine a straightforward mathematical formulation for the luminosity distance as function of the transverse comoving distance for all cosmology cases with a non-zero cosmological constant by adopting a different mindset. The applied method deals with incomplete elliptical integrals of the first kind associated with the polynomial roots admitted in the comoving distance integral according to the scientific literature. The outcome shows that the luminosity distance can be obtained by the combination of an analytical solution followed by a numerical integration in order to account for the redshift. This solution is solely compared to the current Gaussian quadrature method used as basic recognized algorithm in standard cosmology.

Keywords

Cosmology, Distance Luminosity, Transverse Comoving Distance, Incomplete Elliptic Integrals

1. Introduction

Cosmology is a science that relies upon the emitted radiation of the astrophysical sources that we detect with our instruments. Based on that, we measure galactic and cosmological distances with different approaches and mindsets. Moreover, the main task of cosmology consists of making predictions on the description of the physical parameters that we analyze as well as infer statements on the cosmogony. In the cosmological context, analytical and numerical methods play an important role in making predictions, which are accordingly affected by uncertainty and interpretability in order to suit the needs of scientists and various scientific departments. Each method presents benefits and disadvantages which have to be carefully investigated.

The luminosity distance is a tool to measure the cosmological distances and depends on the cosmological model considered. It provides us information about how the radiation faintness of distant astronomical objects appears from our perspective. In the case of the ΛCDM-FLRW (Lambda Cold Dark Matter based on the Friedmann-Lemaitre-Robertson-Walker metric) physical and mathematical frame, the distance luminosity depends on the omega density parameters as a result of Friedmann's approach and equations in the hypothesis of and isotropic and homogeneous Universe. Obviously, we are discussing a 4-dimensional space-time geometry in accordance with the scientific literature of the standard model.

1.1. Existing Methods

An accurate method of calculation of the luminosity distance allows us to test the ΛCDM model and compare it with existing computational methods. Current cosmology adopts the Gaussian quadrature algorithms as well as Romberg's integration to solve the comoving distance integral. The comoving distance is a mathematical parameter that provides us with the current position of an astronomical object in our current epoch and from the terrestrial perspective. However, in the literature, we can find several articles that provide different analytical and numerical solutions to the problem. An analytical solution has already been provided in a different mathematical framework by using Legendre's elliptic integral in a flat cosmology [1] followed by the same analysis for a non-flat cosmology [2] which has not yet been implemented in the scientific community. A numerical method [3] proposes the Carlson symmetric forms which characterize a calculation algorithm, by introducing a change of variable in the luminosity distance formula and by defining a specific elliptic integral as a solution. The method proposed reaches full convergence after iterative computations. The same paper proposes another resolutive method which consists of the approximation by a modified Hermite interpolation. It introduces a new mathematical function as well as a third-order polynomial as linear combination of so-called Hermite basis splines. These fitting algorithms follow similar approaches available in the scientific literature and undertaken by other authors [4] [5]. A method based on the Padè approximant [6] calculates an analytical approximation of the luminosity distance which can also be expressed through an elliptic integral as previously mentioned [2]. Other more complex proceedings infer the luminosity distance by means of the so-called HPM (Homotopy Perturbation Method) simply by reversing the calculation process from solving an integral to solving a set of non-linear differential equations [7] [8]. On this trail, another solving method [9] uses the PSM (Parker-Sochacki Method) based on a polynomial of different non-linear differential equations. In a good recent resume of all methods [10], the authors investigate the distance modulus at various redshift ranges for different astronomical sources. All methods are then compared with observational data with the best fitting plot containing error levels. The context is named cosmography meant as the study of the kinematic properties of the Universe, very critical against applying Taylor expansion series approaches due to the fact that observational data overcome the limit of the expansion series itself. This alters the expected convergence of the methods.

1.2. Legendre-Jacobi's Elliptic Integrals

Differently from the mentioned papers, this inquiry determines the value of the luminosity distance for increasing redshifts z by involving a specific solution of an incomplete class of elliptic integral of the first kind [11] which leads to a specific solution for our space-time cosmology. Depending on the cosmological case under examination, this analysis considers a quartic or a cubic polynomial inside the luminosity distance integral made up of the roots of the cosmological parameters without any approximation. The roots associated with the cosmological parameters show real and complex numbers. One of them is normally the complex conjugated of a parent one. The peculiarities of the roots allow us to identify a specific elliptic integral and to determine, based on its mathematics, the comoving distance value as function of the redshift z, and in turn the luminosity distance by means the $(1 + z)$ factor. It is important to point out that the management of complex numbers in cosmology, in this specific model of analysis, does not influence the reliability or the correctness of the method as the complex numbers associated with the roots of the fourth or third-grade polynomial at the denominator of the integral cancel out in the calculation procedure. It means that a pure mathematical approach translates into a well-defined physical solution. We are dealing with a Legendre-Jacobi elliptic integrals meant as a class of solutions derived from two different mathematical approaches: on one hand, the Legendre's elliptic integrals which can be considered as mathematical functions associated with the analysis of elliptic curves. This class of functions involve an amplitude and a modulus. On the other hand, Jacobi's elliptic functions can be treated as trigonometric functions adopted in the calculation of solution for differential equations which arise from elliptic integral problems.

1.3. Cosmological Parameters

This undertaken method leads to an exact solution which allows to plot the d_L-z graph for the ΛCDM-FLRW based cosmology. Indeed, the distance luminosity is defined by

$$d_L = (1+z)\,d_{tc}, \tag{1}$$

which is valid only for the ΛCDM cosmological framework resulting in an expanding space. d_{tc} is the transverse comoving distance as function of the comoving distance d_c. Accordingly, we can calculate other important parameters in cosmology such as the angular diameter distance d_A which is defined by geometrical reasonings as the ratio between the transversal size of the galaxy D and the subtended angular size ϑ as follows

$$d_A = \frac{D}{\vartheta} = \frac{\vartheta l_0}{\vartheta} = l_0 = \frac{d_{tc}}{1+z}. \tag{2}$$

Specifically, the angular diameter distance allows us to estimate the distance of an astronomical object at the moment the light was emitted toward us. For this reason, it is smaller in value than the transverse comoving distance. The two terms related to the angular size cancel out and what remains is the transversal distance l_0 based on General Relativity which is, in turn, associated with the redshift z as shown in Equation (2). Furtherly, the angular size of a galaxy is another important parameter and topic and it can be calculated through Equation (3) as

$$\vartheta = \frac{D}{d_A} = \frac{D}{\frac{d_{tc}}{1+z}} = \frac{D}{d_{tc}}(1+z), \tag{3}$$

in which we have to assume an average and reliable transversal size of the galaxy equal to 10kpc in order to fulfill the plot. Concerning the unity of measurement, in order to pass from radians to arcseconds, we have to multiply the right-hand side of the equation by a conversion factor as follows:

$$\vartheta = 206265 \frac{D}{d_{tc}}(1+z). \tag{4}$$

We will see in the tables in the coming paragraphs how we actually convert all distances in *Glyrs* (Giga lightyears) in order to uniform the calculation and to minimize the representation scale on the plots. Going back to Equation (1), General Relativity derives the transverse comoving distance in the FLRW framework as solution of Friedmann equations as follows

$$d_{tc} = \begin{cases} \frac{d_H}{\sqrt{\Omega_{k,0}}} \cdot \sinh\left[\frac{d_c}{d_H}\sqrt{\Omega_{k,0}}\right] & \text{for } \Omega_{k,0} > 0 \text{ (open Universe)} \\ d_c & \text{for } \Omega_{k,0} = 0 \text{ (flat Universe)} \\ \frac{d_H}{\sqrt{|\Omega_{k,0}|}} \cdot \sin\left[\frac{d_c}{d_H}\sqrt{|\Omega_{k,0}|}\right] & \text{for } \Omega_{k,0} < 0 \text{ (close Universe)} \end{cases} \tag{5}$$

In the three different expressions of the same physical parameter of Equation (5), we can find the Hubble distance (valid for $z < 0.1$) which has the following expression

$$d_H = \frac{c}{H_0}, \tag{6}$$

where H_0 is the Hubble constant that we can measure in our epoch and that we will better introduce in the next rows. c is the speed of light in vacuum equal to

$$c = 299792458 \frac{\text{m}}{\text{sec}}. \tag{7}$$

Substituting the expression of the Hubble distance of Equation (6) into Equation (5), we obtain

$$d_{tc} = \begin{cases} \frac{c}{H_0}\frac{1}{\sqrt{\Omega_{k,0}}} \cdot \sinh\left[\frac{H_0}{c}\sqrt{\Omega_{k,0}}d_c\right] & \text{for } \Omega_{k,0} > 0 \text{ (open Universe)} \\ d_c & \text{for } \Omega_{k,0} = 0 \text{ (flat Universe)} \\ \frac{c}{H_0}\frac{1}{\sqrt{|\Omega_{k,0}|}} \cdot \sin\left[\frac{H_0}{c}\sqrt{|\Omega_{k,0}|}d_c\right] & \text{for } \Omega_{k,0} < 0 \text{ (close Universe)} \end{cases} \tag{8}$$

With these premises, the equation of interest from Equation (8) in the standard ΛCDM cosmology for the comoving distance is given by

$$d_c = \frac{c}{H_0}\int_0^z \frac{\mathrm{d}z}{\sqrt{\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}}}. \tag{9}$$

For simplicity, we regularly represent the upper integration limit and the variable of the integral argument with the same variable z. It stands for the redshift. Recalling Equation (6), according to the most current observational data from the Planck telescope [12] the Hubble constant in our current epoch is

$$H_0 = 67.36\frac{\mathrm{km}}{\mathrm{sec}\cdot\mathrm{Mpc}} = 2.183\times10^{-18}\frac{1}{\mathrm{sec}}. \tag{10}$$

Moreover, we can list all other relevant parameters starting from the critical density of the Universe in our current epoch equal to

$$\varrho_{c,0} = \frac{3H_0^2}{8\pi G} = 8.521\times10^{-27}\frac{\mathrm{kg}}{\mathrm{m}^3}. \tag{11}$$

From Equation (11), the omega density parameters describing the characteristic of the Universe are defined as function of the critical density in our current epoch, as follows

$$\Omega_{m,0} = \frac{\varrho_{m,0}}{\varrho_{c,0}} = \frac{\varrho_{m,0}}{\frac{3H_0^2}{8\pi G}} = \left(\frac{8\pi G}{3}\varrho_{m,0}\right)\frac{1}{H_0^2} = 0.315. \tag{12}$$

It is the omega density parameter expressing the matter content in the Universe where $\varrho_{m,0}$ is the matter density of the visible Universe and G is the gravitational constant. We can also write

$$\Omega_{r,0} = \frac{\varrho_{r,0}}{\varrho_{c,0}} = \frac{\varrho_{r,0}}{\frac{3H_0^2}{8\pi G}} = \left(\frac{8\pi G}{3}\varrho_{r,0}\right)\frac{1}{H_0^2} = 9.173\times10^{-5}, \tag{13}$$

which is the omega density parameter associated with the radiation where $\varrho_{r,0}$ is the radiation density of the visible Universe, whereas

$$\Omega_{k,0} = -\frac{kc^2}{R_0 H_0^2} = -\frac{kc^2}{H_0^2} = 0.0007 \pm 0.0019, \tag{14}$$

is the omega density parameter associated with the curvature of the 4-D spacetime geometry conceived in the FLRW metric where k is the curvature and R_0 is the scale factor in our epoch (which has a unitary value for rescaling reasonings). Last but not least, we can write

$$\begin{aligned}\Omega_{\Lambda,0} &= \frac{\varrho_{\Lambda,0}}{\varrho_{c,0}} = \frac{\varrho_{\Lambda,0}}{\frac{3H_0^2}{8\pi G}} = \left(\frac{8\pi G}{3}\varrho_{\Lambda,0}\right)\frac{1}{H_0^2} \\ &= \left(\frac{8\pi G}{3}\frac{\Lambda c^2}{8\pi G}\right)\frac{1}{H_0^2} = \left(\frac{\Lambda c^2}{3}\right)\frac{1}{H_0^2} = 0.685.\end{aligned} \tag{15}$$

The latter is the omega density parameter associated with Einstein's cosmological constant which characterizes the dark energy driving the expansion of space. Once listed all these parameters, we can infer, inverting the expression that cosmologists measured with their methods, respectively, the following set of parameters

$$\varrho_{m,0} = 2.686\times10^{-27}\ \frac{\text{kg}}{\text{m}^3}, \tag{16}$$

$$\varrho_{r,0} = 7.816\times10^{-31}\ \frac{\text{kg}}{\text{m}^3}, \tag{17}$$

$$k = -3.711\times10^{-56}\ \frac{1}{\text{m}^2}, \tag{18}$$

$$\varrho_{\Lambda,0} = 5.837\times10^{-27}\ \frac{\text{kg}}{\text{m}^3}, \tag{19}$$

$$\Lambda = 1.089\times10^{-52}\ \frac{1}{\text{m}^2}, \tag{20}$$

where we conceptually consider $\varrho_{r,0}$ and $\varrho_{\Lambda,0}$ equivalent expression forms in order to standardize the units of measurement. Moreover, for definition, the following cosmological relation has to be verified

$$\sum_{j=1}^{4}\Omega_{j,0} = \Omega_{r,0} + \Omega_{m,0} + \Omega_{k,0} + \Omega_{\Lambda,0} = 1. \tag{21}$$

Substituting Equation (9) in the set of Equation (8), it yields

$$d_{tc} = \begin{cases} \dfrac{c}{H_0}\dfrac{1}{\sqrt{\Omega_{k,0}}}\cdot\sinh\left[\dfrac{H_0}{c}\sqrt{\Omega_{k,0}}\dfrac{c}{H_0}\displaystyle\int_0^z\dfrac{\text{d}z}{\sqrt{\Omega_{r,0}(1+z)^4+\Omega_{m,0}(1+z)^3+\Omega_{k,0}(1+z)^2+\Omega_{\Lambda,0}}}\right] & \text{for } \Omega_{k,0}>0\ (\text{open Universe}) \\ \dfrac{c}{H_0}\displaystyle\int_0^z\dfrac{\text{d}z}{\sqrt{\Omega_{r,0}(1+z)^4+\Omega_{m,0}(1+z)^3+\Omega_{\Lambda,0}}} & \text{for } \Omega_{k,0}=0\ (\text{flat Universe}) \\ \dfrac{c}{H_0}\dfrac{1}{\sqrt{|\Omega_{k,0}|}}\cdot\sin\left[\dfrac{H_0}{c}\sqrt{|\Omega_{k,0}|}\dfrac{c}{H_0}\displaystyle\int_0^z\dfrac{\text{d}z}{\sqrt{\sqrt{\Omega_{r,0}(1+z)^4+\Omega_{m,0}(1+z)^3+\Omega_{k,0}(1+z)^2+\Omega_{\Lambda,0}}}}\right] & \text{for } \Omega_{k,0}<0\ (\text{close Universe}) \end{cases} \tag{22}$$

After that some terms associated with the Hubble distance cancel out, we obtain

$$d_{tc} = \begin{cases} \dfrac{c}{H_0}\dfrac{1}{\sqrt{\Omega_{k,0}}}\cdot\sinh\left[\sqrt{\Omega_{k,0}}\displaystyle\int_0^z\dfrac{\text{d}z}{\sqrt{\Omega_{r,0}(1+z)^4+\Omega_{m,0}(1+z)^3+\Omega_{k,0}(1+z)^2+\Omega_{\Lambda,0}}}\right] & \text{for } \Omega_{k,0}>0\ (\text{open Universe}) \\ \dfrac{c}{H_0}\displaystyle\int_0^z\dfrac{\text{d}z}{\sqrt{\Omega_{r,0}(1+z)^4+\Omega_{m,0}(1+z)^3+\Omega_{\Lambda,0}}} & \text{for } \Omega_{k,0}=0\ (\text{flat Universe}) \\ \dfrac{c}{H_0}\dfrac{1}{\sqrt{|\Omega_{k,0}|}}\cdot\sin\left[\sqrt{|\Omega_{k,0}|}\displaystyle\int_0^z\dfrac{\text{d}z}{\sqrt{\sqrt{\Omega_{r,0}(1+z)^4+\Omega_{m,0}(1+z)^3+\Omega_{k,0}(1+z)^2+\Omega_{\Lambda,0}}}}\right] & \text{for } \Omega_{k,0}<0\ (\text{close Universe}) \end{cases} \tag{23}$$

It is the set of mathematical expressions for the transverse comoving distance for each cosmological scenario of our interest. We will focus on each of them during the different case analyses.

2. Calculations

2.1. Open Non-Flat Cosmology $\Omega_{r,0}$, $\Omega_{m,0}$, $\Omega_{k,0}$, $\Omega_{\Lambda,0}$

In our first case under examination and based on Planck observations, $\Omega_{k,0} > 0$

which corresponds to a slight open Universe in Equation (23), where we extract the first equation of interest for the transverse comoving distance

$$d_{tc}=\frac{c}{H_0}\frac{1}{\sqrt{\Omega_{k,0}}}\cdot\sinh\left[\sqrt{\Omega_{k,0}}\int_0^z\frac{\mathrm{d}z}{\sqrt{\Omega_{r,0}(1+z)^4+\Omega_{m,0}(1+z)^3+\Omega_{k,0}(1+z)^2+\Omega_{\Lambda,0}}}\right]. \quad (24)$$

Alternatively, in order to simplify its mathematical expression, we can write that

$$d_{tc}=\frac{c}{H_0}\frac{1}{\sqrt{\Omega_{k,0}}}\cdot\sinh\left[\sqrt{\Omega_{k,0}}\cdot I_{tc}\right], \quad (25)$$

where we introduced the main integral of the transverse comoving distance I_{tc}, which we know was previously part of the comoving distance, given by

$$I_{tc}=\int_0^z\frac{\mathrm{d}z}{\sqrt{\Omega_{r,0}(1+z)^4+\Omega_{m,0}(1+z)^3+\Omega_{k,0}(1+z)^2+\Omega_{\Lambda,0}}}. \quad (26)$$

As noticed, we do not neglect the contribution given by the omega density of radiation and by the curvature term, both commonly ignored in modern computational methods due to their small values. This topic will actually define the next approach in the next paragraph when we discuss the other cosmological case. Focusing on our current study case, therefore, by developing the binomials with different powers in the square root at the denominator, Equation (26) results in

$$I_{tc}=\int_0^z\frac{\mathrm{d}z}{\sqrt{\Omega_{r,0}\left(z^4+4z^3+6z^2+4z+1\right)+\Omega_{m,0}\left(z^3+3z^2+3z+1\right)+\Omega_{k,0}\left(z^2+2z+1\right)+\Omega_{\Lambda,0}}}, \quad (27)$$

$$I_{tc}=\int_0^z\frac{\mathrm{d}z}{\sqrt{\Omega_{r,0}z^4+4\Omega_{r,0}z^3+6\Omega_{r,0}z^2+4\Omega_{r,0}z+\Omega_{r,0}+\Omega_{m,0}z^3+3\Omega_{m,0}z^2+3\Omega_{m,0}z+\Omega_{m,0}+\Omega_{k,0}z^2+2\Omega_{k,0}z+\Omega_{k,0}+\Omega_{\Lambda,0}}} \quad (28)$$

$$I_{tc}=\int_0^z\frac{\mathrm{d}z}{\sqrt{\Omega_{r,0}z^4+\left(4\Omega_{r,0}+\Omega_{m,0}\right)z^3+\left(6\Omega_{r,0}+3\Omega_{m,0}+\Omega_{k,0}\right)z^2+\left(4\Omega_{r,0}+3\Omega_{m,0}+2\Omega_{k,0}\right)z+\Omega_{r,0}+\Omega_{m,0}+\Omega_{k,0}+\Omega_{\Lambda,0}}} \quad (29)$$

Moreover, due to Equation (21), the denominator of Equation (29) changes into

$$I_{tc}=\int_0^z\frac{\mathrm{d}z}{\sqrt{\Omega_{r,0}z^4+\left(4\Omega_{r,0}+\Omega_{m,0}\right)z^3+\left(6\Omega_{r,0}+3\Omega_{m,0}+\Omega_{k,0}\right)z^2+\left(4\Omega_{r,0}+3\Omega_{m,0}+2\Omega_{k,0}\right)z+1}}. \quad (30)$$

Inserting the values of the omega density parameters of Equations (12), (13), (14) and (15), it yields

$$I_{tc}=\int_0^z\frac{\mathrm{d}z}{\sqrt{9.173\times10^{-5}\times z^4+\left(4\times9.173\times10^{-5}+0.315\right)z^3+\left(6\times9.173\times10^{-5}+3\times0.315+0.0007\right)z^2+\left(4\times9.173\times10^{-5}+3\times0.315+2\times0.0007\right)z+1}} \quad (31)$$

or rather

$$I_{tc}=\int_0^z\frac{\mathrm{d}z}{\sqrt{9.173\times10^{-5}z^4+0.315z^3+0.946z^2+0.947z+1}}. \quad (32)$$

We can identify a quartic polynomial in the square root which admits the following roots (determined by different very reliable computational tools available online such as *Wolfram Mathematica*)

$$\begin{cases} r_1 = -2.297 \\ r_2 = -3430.986 \\ r_3 = -0.353 - 1.121i \\ r_4 = \overline{r_3} = -0.353 + 1.121i \end{cases} \tag{33}$$

where i is the imaginary number and $\overline{r_3}$ is the complex conjugated of r_4. Therefore, the main integral of Equation (32) takes now the form

$$I_{tc} = \int_0^z \frac{\mathrm{d}z}{\sqrt{(z-r_1)(z-r_2)(z-r_3)(z-\overline{r_3})}}. \tag{34}$$

The order of the roots in the parenthesis is not casual but follows the rules of the incomplete elliptic integral of the first order containing two complex roots and one of them complex conjugated [11], *integral 260.00*, in which

$$r_2 < r_1 < z < \infty. \tag{35}$$

Thus, the integral of Equation (34) containing the roots of Equation (33) can be written as

$$I_{tc} = \int_0^z \frac{\mathrm{d}z}{\sqrt{\left[z-(-2.297)\right]\left[z-(-3430.986)\right]\left[z-(-0.353-1.121i)\right]\left[z-(-0.353+1.121i)\right]}}. \tag{36}$$

We refer to that specific integral found in the scientific literature for which the solution is provided by the following expression

$$I_{tc} = g_* \times F(\varphi, k_*)\big|_{[0,z]}, \tag{37}$$

where g_* is a constant associated with further coefficients as function of the polynomial roots whereas the incomplete elliptic integral of the first kind in the interval $[0, z]$ is

$$F(\varphi, k_*)\big|_{[0,z]} = F(\varphi, k_*)\big|_{[r_1,z]} - F(\varphi, k_*)\big|_{[r_1,0]}, \tag{38}$$

and it assumes exactly this kind of expression as the main solution of the elliptic integral in the literature considers only the following integral limits $[r_1, z]$. However, our integral extremes lay in-between values. For this reason, we have to subtract the integral contribution in the interval $[r_1, 0]$. Accordingly, Equation (37) becomes

$$I_{tc} = g_* \left[F(\varphi, k_*)\big|_{[r_1,z]} - F(\varphi, k_*)\big|_{[r_1,0]} \right]. \tag{39}$$

We have to calculate the solution in our desired interval by knowing that

$$g_* = \frac{1}{\sqrt{AB}}, \tag{40}$$

in which the coefficients A and B are computed as follows

$$A = \sqrt{\left[r_1 - \frac{r_3 + \overline{r_3}}{2}\right]^2 - \left[\frac{(r_3 - \overline{r_3})^2}{4}\right]}, \tag{41}$$

$$A = \sqrt{\left[-2.297 - \frac{-0.353 - 1.121i - 0.353 + 1.121i}{2}\right]^2 - \left[\frac{\left(-0.353 - 1.121i - (-0.353 + 1.121i)\right)^2}{4}\right]} \tag{42}$$

which leads to

$$A = 2.244, \tag{43}$$

and

$$B = \sqrt{\left[r_2 - \frac{r_3 + \overline{r}_3}{2}\right]^2 - \left[\frac{\left(r_3 - \overline{r}_3\right)^2}{4}\right]}, \tag{44}$$

$$B = \sqrt{\left[-3430.986 - \frac{-0.353 - 1.121i - 0.353 + 1.121i}{2}\right]^2 - \left[\frac{\left(-0.353 - 1.121i - \left(-0.353 + 1.121i\right)\right)^2}{4}\right]} \tag{45}$$

which is ultimately

$$B = 3430.634. \tag{46}$$

Therefore, Equation 40 becomes

$$g_* = \frac{1}{\sqrt{2.244 \times 3430.634}} = 0.017. \tag{47}$$

The general formulation of the incomplete elliptical integral of the first order, underlining the upper limit l_{up}, is

$$F\left(\varphi, k_*\right)\Big|_{\left[r_1, l_{up}\right]} = \int_0^{\varphi\left[r_1, l_{up}\right]} \frac{\mathrm{d}\theta}{\sqrt{1 - k_*^2 \sin^2\theta}}, \tag{48}$$

where the Jacobi's amplitude is given by

$$\varphi\Big|_{\left[r_1, l_{up}\right]} = \arccos\left[\frac{\left(A - B\right) l_{up} + r_1 B - r_2 A}{\left(A + B\right) l_{up} - r_1 B - r_2 A}\right], \tag{49}$$

whereas the elliptic modulus is

$$k_* = \sqrt{\frac{\left(A + B\right)^2 - \left(r_1 - r_2\right)^2}{4AB}}. \tag{50}$$

We can start from the calculation of the latter, as k_* has the same value for both intervals in the elliptic integrals $F\left(\varphi, k_*\right)\Big|_{\left[r_1, z\right]}$ and $F\left(\varphi, k_*\right)\Big|_{\left[r_1, 0\right]}$. Therefore,

$$k_* = \sqrt{\frac{\left(2.244 + 3430.634\right)^2 - \left[-2.297 - \left(-3430.986\right)\right]^2}{4 \times 2.244 \times 3430.634}} = 0.966. \tag{51}$$

It is a valid result as a condition to verify is

$$-1 < k_* < 1. \tag{52}$$

Based on the logic that we previously discussed in Equation (38), we can start from $F\left(\varphi, k_*\right)\Big|_{\left[r_1, z\right]}$ so that $l_{up} = z$. Based on the scientific literature, we have to first evaluate

$$\varphi\Big|_{\left[r_1, z\right]} = \arccos\left[\frac{\left(A - B\right) z + r_1 B - r_2 A}{\left(A + B\right) z - r_1 B - r_2 A}\right], \tag{53}$$

$$\varphi\Big|_{\left[r_1, z\right]} = \arccos\left[\frac{\left(2.244 - 3430.634\right) z + \left(-2.297 \times 3430.634\right) - \left(-3430.986 \times 2.244\right)}{\left(2.244 + 3430.634\right) z - \left(-2.297 \times 3430.634\right) - \left(-3430.986 \times 2.244\right)}\right], \tag{54}$$

which eventually leads to

$$\varphi\big|_{[r_1,z]} = \arccos\left(\frac{-3428.39z - 185.123}{3432.878z + 15583.387}\right). \tag{55}$$

This expression is exactly responsible for the request of a numerical method as integration of the analytical one. Therefore, the first incomplete elliptic integral of the first order of Equation (48) in the interval $[r_1, z]$ is given by

$$F(\varphi, k_*)\big|_{[r_1,z]} = \int_0^{\varphi|_{[r_1,z]}} \frac{\mathrm{d}\theta}{\sqrt{1-k_*^2\sin^2\theta}}, \tag{56}$$

$$F(\varphi, k_*)\big|_{[r_1,z]} = \int_0^{\arccos\left(\frac{-3428.39z-185.123}{3432.878z+15583.387}\right)} \frac{\mathrm{d}\theta}{\sqrt{1-0.966^2\sin^2\theta}}, \tag{57}$$

$$F(\varphi, k_*)\big|_{[r_1,z]} = \int_0^{\arccos\left(\frac{-3428.39z-185.123}{3432.878z+15583.387}\right)} \frac{\mathrm{d}\theta}{\sqrt{1-0.933\sin^2\theta}}. \tag{58}$$

Moreover, considering the remaining incomplete elliptic integral $F(\varphi, k_*)\big|_{[r_1,0]}$ with $l_{up} = 0$, it yields

$$\varphi\big|_{[r_1,0]} = \arccos\left[\frac{(A-B)0 + r_1B - r_2A}{(A+B)0 - r_1B - r_2A}\right], \tag{59}$$

$$\varphi\big|_{[r_1,0]} = \arccos\left[\frac{(-2.297\times 3430.634) - (-3430.986\times 2.244)}{-(-2.297\times 3430.634) - (-3430.986\times 2.244)}\right] = 1.583\ \text{rad}. \tag{60}$$

Therefore, the second incomplete elliptic integral of the first order of Equation (48) in the interval $[r_1, 0]$ is provided by

$$F(\varphi, k_*)\big|_{[r_1,0]} = \int_0^{\varphi|_{[r_1,0]}} \frac{\mathrm{d}\theta}{\sqrt{1-k_*^2\sin^2\theta}}, \tag{61}$$

$$F(\varphi, k_*)\big|_{[r_1,0]} = \int_0^{1.583} \frac{\mathrm{d}\theta}{\sqrt{1-0.966^2\sin^2\theta}} = 2.8151. \tag{62}$$

If we step back to the expression main integral of the transverse comoving distance of Equation (25), in order to determine the value of the incomplete elliptic integral of the first order in the interval $[0, z]$ in Equation (39), we can write that

$$I_{tc} = 0.017\times\left[\left(\int_0^{\arccos\left(\frac{-3428.39z-185.123}{3432.878z+15583.387}\right)} \frac{\mathrm{d}\theta}{\sqrt{1-0.933\sin^2\theta}}\right) - 2.8151\right]. \tag{63}$$

Therefore, Equation (25) expressed in meters, eventually becomes

$$d_{tc} = \frac{299792458}{2.183\times 10^{-18}}\frac{1}{\sqrt{0.0007}}\times\sinh\left\{\sqrt{0.0007}\times 0.017\times\left[\left(\int_0^{\arccos\left(\frac{-3428.39z-185.123}{3432.878z+15583.387}\right)} \frac{\mathrm{d}\theta}{\sqrt{1-0.933\sin^2\theta}}\right) - 2.8151\right]\right\}, \tag{64}$$

$$d_{tc} = 5.19\times 10^{27}\times\sinh\left\{4.497\times 10^{-4}\times\left[\left(\int_0^{\arccos\left(\frac{-3428.39z-185.123}{3432.878z+15583.387}\right)} \frac{\mathrm{d}\theta}{\sqrt{1-0.933\sin^2\theta}}\right) - 2.8151\right]\right\}. \tag{65}$$

In order to evaluate the transverse comoving distance, we have to consider numerically different values of z in order to determine the integral.

2.2. Open Non-Flat Cosmology $\Omega_{m,0}$, $\Omega_{k,0}$, $\Omega_{\Lambda,0}$

Compared to the previous case, we consider in this scenario the following assumption

$$\Omega_{r,0} \cong 0, \tag{66}$$

which is justified by the small observational value and it is basically the main hypothesis made by Carroll [13]. Similarly, $\Omega_{k,0} > 0$ which corresponds to a slight open Universe. Basically, we are now dealing with one omega density parameter less but we are similarly involving the same mathematics and physics of an open Universe. Based on that, due to Equation (25) the integral of Equation (26) becomes

$$I_{tc} = \int_0^z \frac{\mathrm{d}z}{\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}}}. \tag{67}$$

As in this case Equation (21) has one parameter less, it yields

$$\sum_{j=1}^{3} \Omega_{j,0} = \Omega_{m,0} + \Omega_{k,0} + \Omega_{\Lambda,0} = 1, \tag{68}$$

from which we can write that

$$\Omega_{k,0} = 1 - \Omega_{\Lambda,0} - \Omega_{m,0}. \tag{69}$$

According to some in-between algebraic steps, we can exactly reach Carroll's formula [13] as follows

$$I_{tc} = \int_0^z \frac{\mathrm{d}z}{\sqrt{\Omega_{m,0}(1+z)^3 + (1 - \Omega_{\Lambda,0} - \Omega_{m,0})(1+z)^2 + \Omega_{\Lambda,0}}}, \tag{70}$$

$$I_{tc} = \int_0^z \frac{\mathrm{d}z}{\sqrt{\Omega_{m,0}(1+z)^3 + (1+z)^2(1 - \Omega_{m,0}) - \Omega_{\Lambda,0}\left[(1+z)^2 - 1\right]}}, \tag{71}$$

$$I_{tc} = \int_0^z \frac{\mathrm{d}z}{\sqrt{(1+z)^2(1 + \Omega_{m,0}z) - z(2+z)\Omega_{\Lambda,0}}}. \tag{72}$$

However, our aim is to continue with the algebraic steps in Equation (72) as we want to obtain a polynomial in the variable z with a certain grade in order to be able to discuss the corresponding incomplete elliptic integral. Therefore, substituting the values for $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$, respectively of Equation (12) and Equation (15), in Equation (72), it yields

$$I_{tc} = \int_0^z \frac{\mathrm{d}z}{\sqrt{(z^2 + 2z + 1)(1 + 0.315z) - z(2+z)0.685}}, \tag{73}$$

which leads to

$$I_{tc} = \int_0^z \frac{\mathrm{d}z}{\sqrt{0.315z^3 + 0.945z^2 + 0.945z + 1}}. \tag{74}$$

This time, we recognize a cubic polynomial in the square root at the denominator which admits the following roots

$$\begin{cases} r_1 = -2.295 \\ r_2 = -0.352 - 1.122i \\ r_3 = \overline{r_2} = -0.352 + 1.122i \end{cases} \tag{75}$$

where i is the imaginary number and $\overline{r_2}$ is the complex conjugated of r_2. Therefore, the integral of Equation (74) takes now the form

$$I_{tc} = \int_0^z \frac{dz}{\sqrt{(z - r_1)(z - r_2)(z - \overline{r_2})}}, \tag{76}$$

$$I_{tc} = \int_0^z \frac{dz}{\sqrt{[z - (-2.295)][z - (-0.352 - 1.122i)][z - (-0.352 + 1.122i)]}}, \tag{77}$$

which leads eventually to

$$I_{tc} = \int_0^z \frac{dz}{\sqrt{[z - (-2.295)]\{[z - (-0.352)]^2 + 1.259\}}}. \tag{78}$$

Therefore, the expression of the transverse comoving distance of Equation (25) is now

$$d_{tc} = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{k,0}}} \times \sinh\left[\sqrt{\Omega_{k,0}} \int_0^z \frac{dz}{\sqrt{[z - (-2.295)]\{[(z - (-0.352)]^2 + 1.259\}}}\right]. \tag{79}$$

Through Equation (78), we obtained exactly the formulation of the incomplete elliptic integral of the first kind [11], this time corresponding in the literature to *integral 239.00*, where we can infer, according to the integral terminology, that its coefficients, which will be used for the calculation of the parameters, are the following

$$b_1 = -0.352, \tag{80}$$

and

$$a_1^2 = 1.259. \tag{81}$$

The integral admits the same type of solution of Equation (39) with the same reasoning concerning the interval calculations. However, due to the new incomplete elliptic integral of the first kind under examination, we have to calculate the solution in our desired interval by knowing that this time

$$g_* = \frac{1}{\sqrt{A}}, \tag{82}$$

in which the coefficient A can be computed according to the literature as follows

$$A = \sqrt{[b_1 - r_1]^2 + a_1^2}, \tag{83}$$

which leads to

$$A = \sqrt{[-0.352 - (-2.295)]^2 + 1.259} = 2.244. \tag{84}$$

From this result, which is the same calculated in the non-flat cosmology case, we can calculate in Equation (82) that

$$g_* = \frac{1}{\sqrt{2.244}} = 0.667. \tag{85}$$

The incomplete elliptical integral of the first order, underlining the upper limit l_{up}, is given by Equation (48) where we know identify different intrinsic parameters of the integral such as the Jacobis amplitude given by

$$\varphi\big|_{[r_1,l_{up}]} = \arccos\left[\frac{A+r_1-l_{up}}{A-r_1+l_{up}}\right], \tag{86}$$

whereas the elliptic modulus is

$$k_* = \sqrt{\frac{A+b_1-r_1}{2A}}. \tag{87}$$

As previously done, we can start from the calculation of the latter, as k_* has the same value for both intervals in the elliptic integrals $F(\varphi,k_*)\big|_{[r_1,z]}$ and $F(\varphi,k_*)\big|_{[r_1,0]}$. Therefore,

$$k_* = \sqrt{\frac{2.244-0.352-(-2.295)}{2\times 2.244}} = 0.966. \tag{88}$$

Despite we are dealing with different coefficients, also in this case we calculated the same elliptic modulus which verifies the condition Equation (52). Based on the logic that we previously discussed, we can start from $F(\varphi,k_*)\big|_{[r_1,z]}$ in Equation (86) so that $l_{up} = z$. It yields

$$\varphi\big|_{[r_1,z]} = \arccos\left[\frac{A+r_1-z}{A-r_1+z}\right], \tag{89}$$

$$\varphi\big|_{[r_1,z]} = \arccos\left[\frac{2.244-2.295-z}{2.244-(-2.295)+z}\right], \tag{90}$$

$$\varphi\big|_{[r_1,z]} = \arccos\left(\frac{-0.051-z}{4.539+z}\right). \tag{91}$$

Due to Equation (91), the first incomplete elliptic integral of the first kind in the interval $[r_1,z]$ of Equation (56) is given by

$$F(\varphi,k_*)\big|_{[r_1,z]} = \int_0^{\arccos\left(\frac{-0.051-z}{4.539+z}\right)} \frac{\mathrm{d}\theta}{\sqrt{1-0.933\sin^2\theta}}. \tag{92}$$

Additionally, considering the remaining incomplete elliptic integral $F(\varphi,k_*)\big|_{[r_1,0]}$ from Equation (86) with $l_{up} = 0$. It yields

$$\varphi\big|_{[r_1,0]} = \arccos\left[\frac{A+r_1-0}{A-r_1+0}\right]. \tag{93}$$

It leads to

$$\varphi\big|_{[r_1,0]} = \arccos\left[\frac{2.244-2.295}{2.244-(-2.295)}\right] = 1.582\ \text{rad}. \tag{94}$$

Therefore, the second incomplete elliptic integral of the first order in the interval $\left[r_1, 0\right]$ of Equation (61) is provided by

$$F(\varphi, k_*)\big|_{[r_1,0]} = \int_0^{1.582} \frac{d\theta}{\sqrt{1-0.933\sin^2\theta}} = 2.811. \quad (95)$$

In order to determine the value of the incomplete elliptic integral of the first order in the interval $\left[0, z\right]$, we can write that Equation (39) becomes

$$I_{tc} = 0.667 \times \left[\left(\int_0^{\arccos\left(\frac{-0.051-z}{4.539+z}\right)} \frac{d\theta}{\sqrt{1-0.933\sin^2\theta}}\right) - 2.811\right]. \quad (96)$$

If we step back to Equation (25), the expression main integral of the transverse comoving distance in meters becomes now

$$d_{tc} = \frac{299792458}{2.183\times10^{-18}} \frac{1}{\sqrt{0.0007}} \times \sinh\left\{\sqrt{0.0007} \times 0.667 \times \left[\left(\int_0^{\arccos\left(\frac{-0.051-z}{4.539+z}\right)} \frac{d\theta}{\sqrt{1-0.933\sin^2\theta}}\right) - 2.811\right]\right\}, \quad (97)$$

$$d_{tc} = 5.19\times10^{27} \times \sinh\left\{0.017 \times \left[\left(\int_0^{\arccos\left(\frac{-0.051-z}{4.539+z}\right)} \frac{d\theta}{\sqrt{1-0.933\sin^2\theta}}\right) - 2.811\right]\right\}. \quad (98)$$

Also in this case, it is necessary to add a numerical analysis to the analytical one, in order to evaluate the transverse comoving distance at each redshift.

2.3. Closed Non-Flat Cosmology $\Omega_{r,0}$, $\Omega_{m,0}$, $\Omega_{k,0}$, $\Omega_{\Lambda,0}$

A closed Universe implies $\Omega_{k,0} < 0$ which can be obtained, for instance, subtracting the negative tolerance values from the omega curvature parameter in Equation (14) as follows

$$\Omega_{k,0} = 0.0007 - 0.0019 = -0.0012. \quad (99)$$

In this cosmological case, we extract the third equation from the set of Equation (23)

$$d_{tc} = \frac{c}{H_0} \frac{1}{\sqrt{|\Omega_{k,0}|}} \times \sin\left[\sqrt{|\Omega_{k,0}|} \times I_{tc}\right], \quad (100)$$

We are dealing with four omega density parameters which will surely ensure a quartic polynomial in the expression at the denominator of the integral of Equation (26). Accordingly, substituting the new obtained value of Equation (99) into previous Equation (26), it yields

$$I_{tc} = \int_0^z \frac{dz}{\sqrt{9.173\times10^{-5}\times z^4 + \left(4\times9.173\times10^{-5}+0.315\right)z^3 + \left(6\times9.173\times10^{-5}+3\times0.315-0.0012\right)z^2 + \left(4\times9.173\times10^{-5}+3\times0.315-2\times0.0012\right)z+1}} \quad (101)$$

or rather

$$I_{tc} = \int_0^z \frac{dz}{\sqrt{9.173\times10^{-5}z^4 + 0.315z^3 + 0.944z^2 + 0.943z + 1}}. \quad (102)$$

The quartic polynomial in the square root admits the following roots

$$\begin{cases} r_1 = -2.297 \\ r_2 = -3430.99 \\ r_3 = -0.351 - 1.122i \\ r_4 = \overline{r_3} = -0.351 + 1.122i \end{cases} \quad (103)$$

where i is the imaginary number and $\overline{r_3}$ is the complex conjugated of r_4.

Therefore, the main integral takes the same form of Equation (34). Moreover, we recognize once again the *integral 260.00* [11], where the condition of Equation (35) is also verified. Thus, we can explicit the main integral I_{tc} of Equation (34) as

$$I_{tc} = \int_0^z \frac{\mathrm{d}z}{\sqrt{\left[z-(-2.297)\right]\left[z-(-3430.99)\right]\left[z-(-0.351-1.122i)\right]\left[z-(-0.351+1.122i)\right]}}. \quad (104)$$

The procedure is identical to the procedure undergone in a non-flat open Universe. However, the values in outcome are not identical. The solution of the main integral is given by Equation (37) in which the incomplete elliptic integral of the first kind in the interval $[0,z]$ is provided by Equation (38). The constant g_* is calculated according to Equation (40). Its coefficients A and B have the same mathematical expressions, respectively, coming from Equation (41) and Equation (44). Going into detail with their calculation, we obtain that

$$A = \sqrt{\left[-2.297 - \frac{-0.351-1.122i-0.351+1.122i}{2}\right]^2 - \left[\frac{\left(-0.351-1.122i-(-0.351+1.122i)\right)^2}{4}\right]} \quad (105)$$

which leads to

$$A = 2.246, \quad (106)$$

as well as

$$B = \sqrt{\left[-3430.99 - \frac{-0.351-1.122i-0.351+1.122i}{2}\right]^2 - \left[\frac{\left(-0.351-1.122i-(-0.351+1.122i)\right)^2}{4}\right]} \quad (107)$$

which ends up with

$$B = 3430.639. \quad (108)$$

Therefore, from Equation (40), we can calculate that

$$g_* = \frac{1}{\sqrt{2.246 \times 3430.639}} = 0.0114. \quad (109)$$

The incomplete elliptical integral of the first order, underlining the upper limit l_{up}, has been already introduced in Equation (48) as well as the Jacobis amplitude in Equation (49) and the elliptic modulus in Equation (50). From the latter, we can calculate that

$$k_* = \sqrt{\frac{(2.246+3430.639)^2 - \left[-2.297-(-3430.99)\right]^2}{4 \times 2.246 \times 3430.639}} = 0.966. \quad (110)$$

It verifies the condition of Equation (52) and starting with the same calculation logic, from the interval $[r_1, z]$, we can write from Equation (53) that

$$\varphi\big|_{[r_1,z]} = \arccos\left[\frac{(2.246-3430.639)z + (-2.297 \times 3430.639) - (-3430.99 \times 2.246)}{(2.246+3430.639)z - (-2.297 \times 3430.639) - (-3430.99 \times 2.246)}\right] \quad (111)$$

which leads to

$$\varphi\big|_{[r_1,z]} = \arccos\left(\frac{-3428.393z - 174.174}{3432.885z + 15586.18}\right). \quad (112)$$

Due to this, the first incomplete elliptic integral of the first order in the interval $\left[r_1, z\right]$ given by Equation (56), can be calculated as

$$F\left(\varphi, k_*\right)\Big|_{\left[r_1, z\right]} = \int_0^{\arccos\left(\frac{-3428.393z-174.174}{3432.885z+15586.18}\right)} \frac{\mathrm{d}\theta}{\sqrt{1-0.933\sin^2\theta}}. \tag{113}$$

Moreover, considering the remaining incomplete elliptic integral $F\left(\varphi, k_*\right)\Big|_{\left[r_1, 0\right]}$ with $l_{up} = 0$ in Equation (59), it yields

$$\varphi\Big|_{\left[r_1, 0\right]} = \arccos\left[\frac{\left(-2.297 \times 3430.639\right) - \left(-3430.99 \times 2.246\right)}{-\left(-2.297 \times 3430.639\right) - \left(-3430.99 \times 2.246\right)}\right] = 1.582 \text{ rad} \tag{114}$$

Therefore, the second incomplete elliptic integral of the first order in the interval $\left[r_1, 0\right]$ in Equation (61) is provided by

$$F\left(\varphi, k_*\right)\Big|_{\left[r_1, 0\right]} = \int_0^{1.582} \frac{\mathrm{d}\theta}{\sqrt{1-0.933\sin^2\theta}} = 2.811. \tag{115}$$

If we step back to the expression of the main integral of the transverse comoving distance of Equation (37), in order to determine the value of the incomplete elliptic integral of the first order in the interval $\left[0, z\right]$, we can write that

$$I_{tc} = 0.0114 \times \left[\left(\int_0^{\arccos\left(\frac{-3428.393z-174.174}{3432.885z+15586.18}\right)} \frac{\mathrm{d}\theta}{\sqrt{1-0.933\sin^2\theta}}\right) - 2.811\right]. \tag{116}$$

Therefore, Equation (100), expressed in meters, becomes

$$d_{tc} = \frac{299792458}{2.183 \times 10^{-18}} \frac{1}{\sqrt{\left|-0.0012\right|}} \times \sin\left\{\sqrt{\left|-0.0012\right|} \times 0.0114 \times \left[\left(\int_0^{\arccos\left(\frac{-3428.393z-174.174}{3432.885z+15586.18}\right)} \frac{\mathrm{d}\theta}{\sqrt{1-0.933\sin^2\theta}}\right) - 2.811\right]\right\}, \tag{117}$$

$$d_{tc} = 3.96 \times 10^{27} \times \sin\left\{3.95 \times 10^{-4} \times \left[\left(\int_0^{\arccos\left(\frac{-3428.393z-174.174}{3432.885z+15586.18}\right)} \frac{\mathrm{d}\theta}{\sqrt{1-0.933\sin^2\theta}}\right) - 2.811\right]\right\}. \tag{118}$$

A numerical analysis is essential to complete the analytical calculation for the transverse comoving distance.

2.4. Flat Cosmology $\Omega_{m,0}$, $\Omega_{\Lambda,0}$

In this special study case, which is very common in the scientific literature, the comoving distance coincides with the transverse comoving distance in the second equation of the set Equation (23). Due to that,

$$d_{tc} \equiv d_c = \frac{c}{H_0} \int_0^z \frac{\mathrm{d}z}{\sqrt{\Omega_{r,0}\left(1+z\right)^4 + \Omega_{m,0}\left(1+z\right)^3 + \Omega_{k,0}\left(1+z\right)^2 + \Omega_{\Lambda,0}}}. \tag{119}$$

However, precisely because we are dealing with a flat cosmology, we can neglect the following omega density parameters

$$\Omega_{k,0} = \Omega_{r,0} \cong 0, \tag{120}$$

and therefore, Equation (9) assumes the following expression

$$d_{tc} = \frac{c}{H_0} \int_0^z \frac{\mathrm{d}z}{\sqrt{\Omega_{m,0}\left(1+z\right)^3 + \Omega_{\Lambda,0}}}. \tag{121}$$

Due to this, the previous relation of Equation (21) shows only the sum of two single contributions

$$\sum_{j=1}^{2}\Omega_{j,0}=\Omega_{m,0}+\Omega_{\Lambda,0}=1, \tag{122}$$

from which, we can write that

$$\Omega_{\Lambda,0}=1-\Omega_{m,0}. \tag{123}$$

Thus, we can express the argument of the square root at the denominator as function only of the omega matter density parameter. Equation (121) reduces to

$$d_{tc}=\frac{c}{H_0}\int_0^z\frac{\mathrm{d}z}{\sqrt{\Omega_{m,0}(1+z)^3+1-\Omega_{m,0}}}, \tag{124}$$

$$d_{tc}=\frac{c}{H_0}\int_0^z\frac{\mathrm{d}z}{\sqrt{\Omega_{m,0}\left[(1+z)^3-1\right]+1}}, \tag{125}$$

$$d_{tc}=\frac{c}{H_0}\int_0^z\frac{\mathrm{d}z}{\sqrt{\Omega_{m,0}\left[z^3+3z^2+3z\right]+1}}. \tag{126}$$

It is only function of the omega density parameter for matter, which has the value of Equation (12), leading to

$$d_{tc}=\frac{c}{H_0}\int_0^z\frac{\mathrm{d}z}{\sqrt{0.315z^3+0.945z^2+0.945z+1}}. \tag{127}$$

We recognize exactly Equation (74) in a non-flat open cosmology. Because of that, we can copy all the coefficients calculated in the previous cosmological case. We consider valid the results of Equation (85) for g_*, Equation (84) for *A*, Equation (88) for k_* and Equation (96) for the main integral I_{tc}. However, the final result provided by the transverse comoving distance is different from a non-flat open cosmology as there is no more the operator *sinh* in the formulation. Therefore, Equation (127), expressed in meters, becomes

$$d_{tc}=\frac{299792458}{2.183\times10^{-18}}0.667\times\left\{\left[\int_0^{\arccos\left(\frac{-0.051-z}{4.539+z}\right)}\frac{\mathrm{d}\theta}{\sqrt{1-0.933\sin^2\theta}}\right]-2.8112\right\}, \tag{128}$$

or rather

$$d_{tc}=9.16\times10^{25}\times\left\{\left[\int_0^{\arccos\left(\frac{-0.051-z}{4.539+z}\right)}\frac{\mathrm{d}\theta}{\sqrt{1-0.933\sin^2\theta}}\right]-2.8112\right\}. \tag{129}$$

Also in this cosmological case, in order to evaluate the first incomplete elliptic integral of the first order in the parenthesis, we have to consider numerically different values of z in order to determine the integral. Doing this, we can calculate the transverse comoving distance at each redshift and, in turn, the luminosity distance. The plot of the transverse comoving distance and the luminosity distance will be shown in the next chapter.

3. Graphs and Calculation Sheets

In the following graphs, we will plot the predictions of the most important cosmological factors (transverse comoving distance, luminosity distance, angular diameter distance and angular size) in the ΛCDM-FLRW model according to the Gaussian quadrature numerical method (GAUSS-Q, blue curve) compared to a solution based on Legendre-Jacobi's incomplete elliptic integral from Byrd-Friedman's handbook (*int. 260.00*) of the first kind in an open non-flat cosmology with all omega parameters (IEI-1K, brown curve), in an open non-flat cosmology with the radiation contribution negligible (IEI-1K, green curve) (*int. 239.00*), in a closed non-flat cosmology (IEI-1K, violet curve) (*int. 260.00*) and in a flat cosmology (IEI-1K, orange curve) (*int. 239.00*). As previously discussed, the reference integral for implementing the method depends on the number of roots that the polynomial admits. In turn, it depends on the assumptions made concerning the omega density parameters in the equations. For simplicity, we denote with the abbreviation IEI-1K the incomplete elliptic integral of the first kind as well as the abbreviation GAUSS-Q for the numerical computational method named Gaussian quadrature, not covered in this analysis, but largely used in cosmology for the calculation of the transverse comoving distance.

3.1. Transverse Comoving Distance

Starting exactly from the transverse comoving distance, in order to evaluate the first incomplete elliptic integral of the first order in the parenthesis, we have to consider numerically different values of z in order to determine the integral. In this way, we can calculate a specific distance at each redshift and, in turn, also the luminosity distance. The plot of the transverse comoving distance is shown in **Figure 1**.

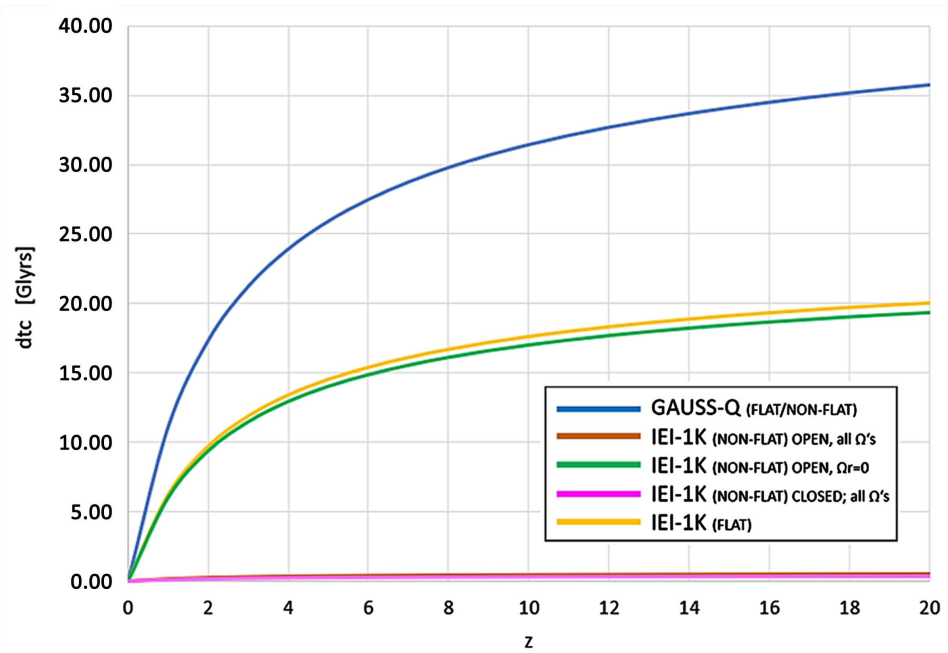

Figure 1. Predictions of the transverse comoving distance.

As two curves overlap in the down part of the plot for small values of distance, we can focus on them by reducing the distance scale in the y axis, as shown in Figure 2.

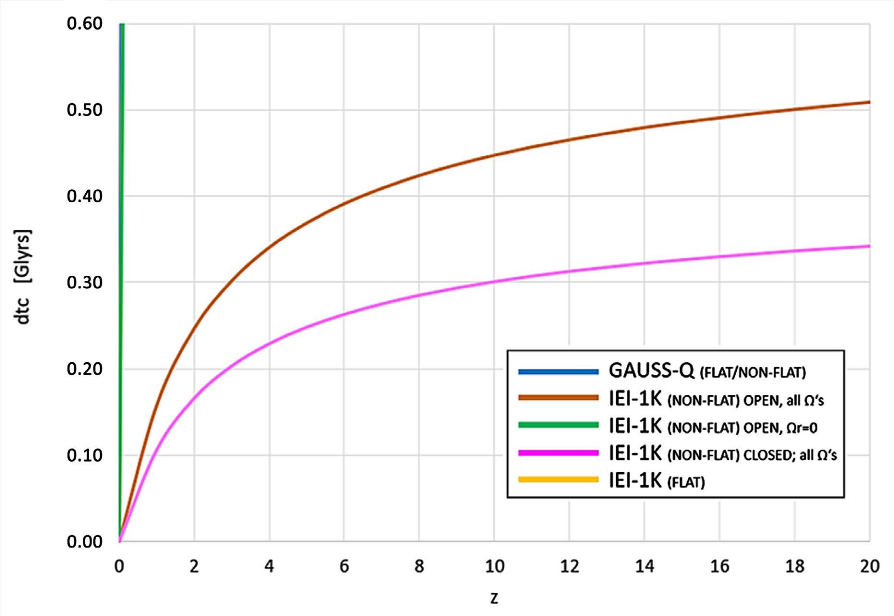

Figure 2. Predictions of the transverse comoving distance. It is the plot of Figure 1 with a smaller scale in order to highlight the two curves at the bottom previously overlapping.

3.2. Luminosity Distance

Once calculated the transverse comoving distance at each redshift, the luminosity distance of Equation (1) is represented by the plot in Figure 3.

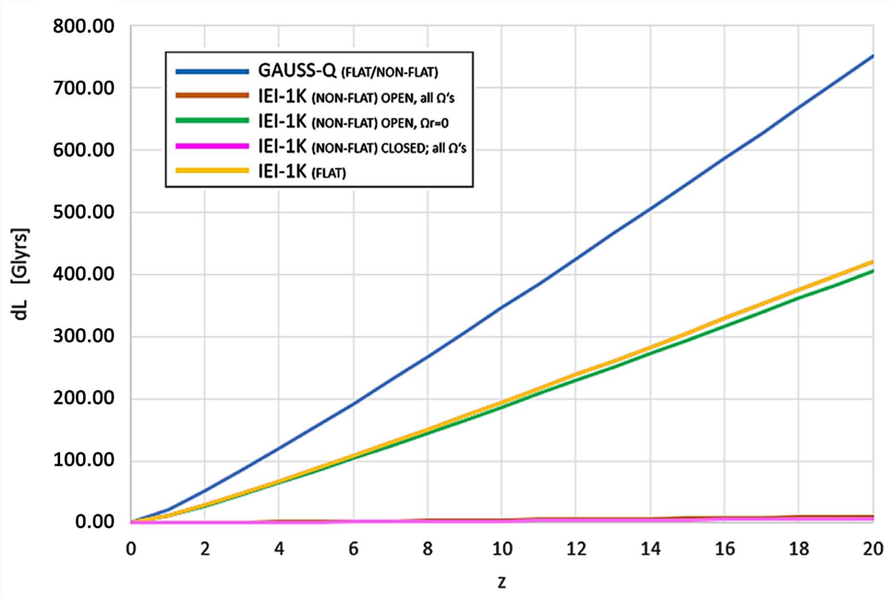

Figure 3. Predictions of the luminosity distance.

Also in this case, two curves overlap and a reduction of the distance scale on the y axis is require in order to distinguish their values. The related plot is shown in Figure 4.

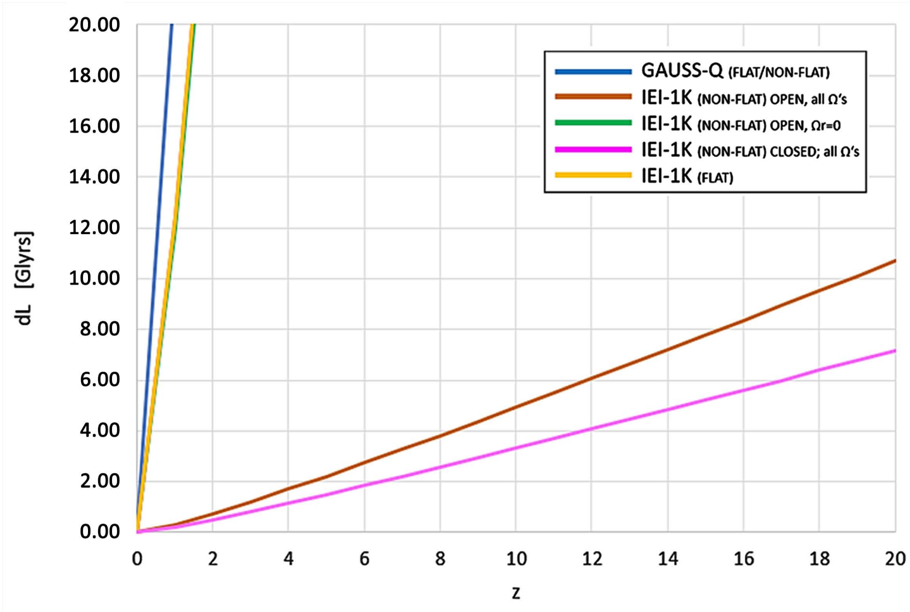

Figure 4. Predictions of the luminosity distance. It is the plot of Figure 3 with a smaller scale in order to highlight the two curves at the bottom previously overlapping.

3.3. Angular Diameter Distance

The angular diameter distance of Equation (2) is shown in Figure 5.

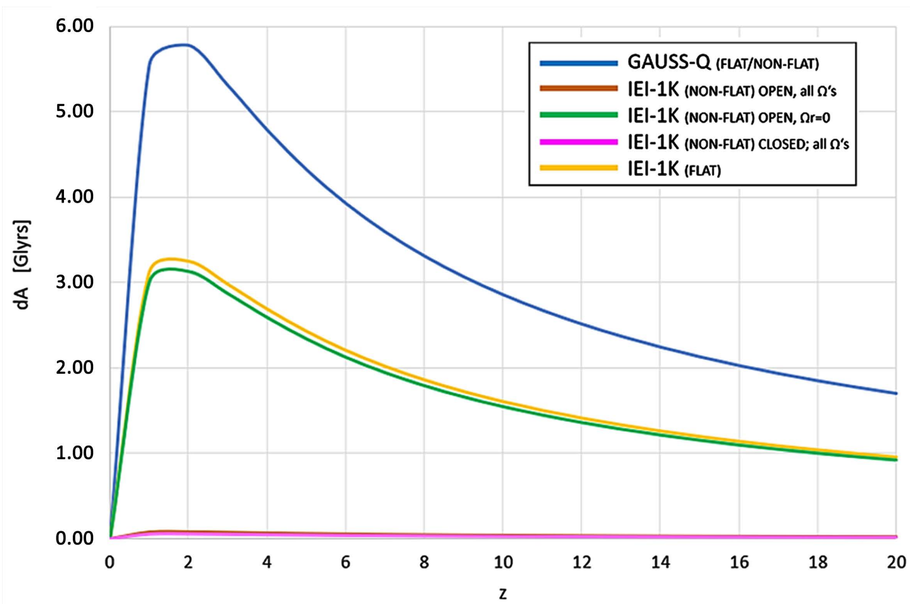

Figure 5. Predictions of the angular diameter distance.

Similar to previous reasoning, we reduce the distance scale and we obtain Figure 6.

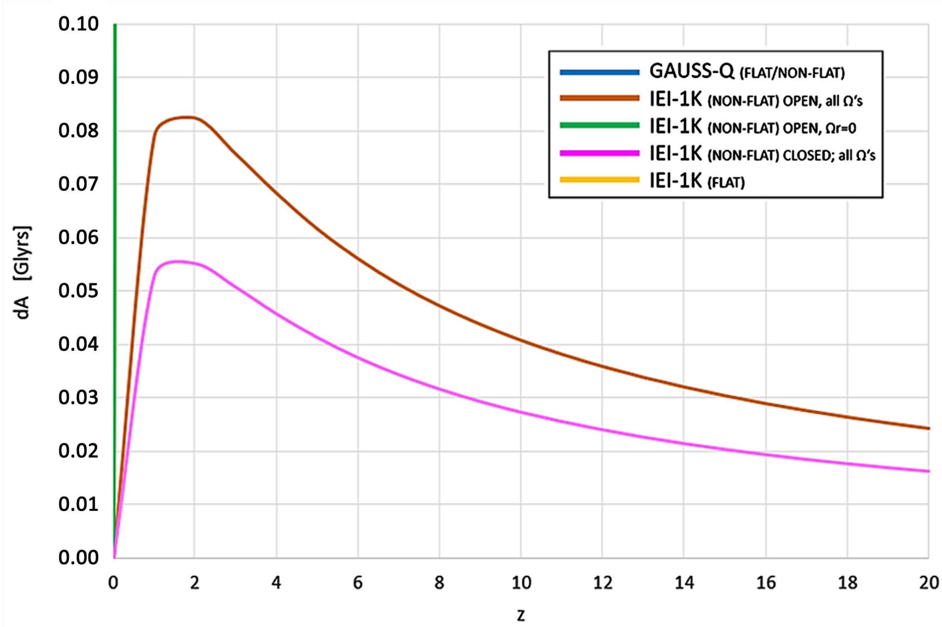

Figure 6. Predictions of the angular diameter distance. It is the plot of **Figure 5** with a smaller scale in order to highlight the two curves at the bottom previously overlapping.

3.4. Angular Size

The plot of the angular size associated with Equation (3) is shown in **Figure 7**. It decreases down to a minimum for than increasing again and it is one of the most important characteristics of standard cosmology.

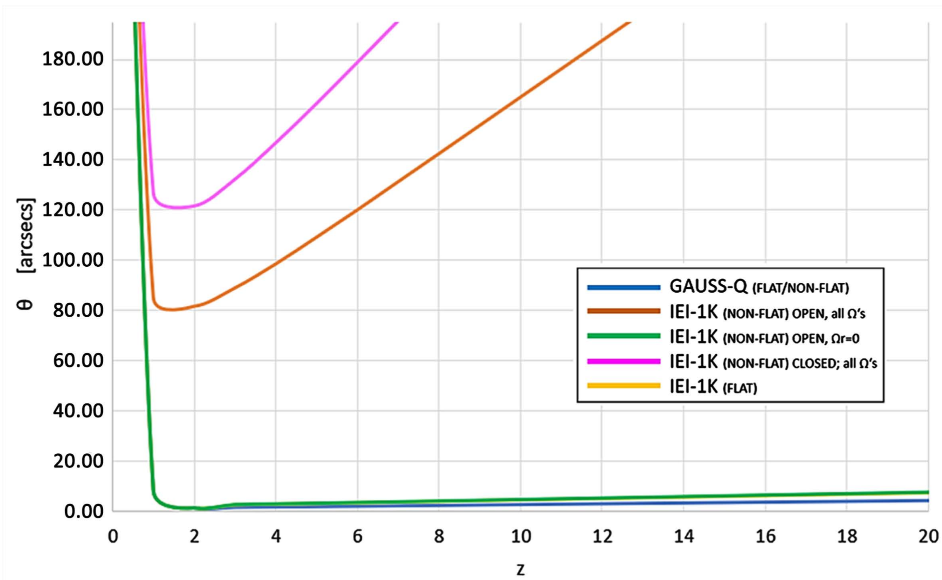

Figure 7. Predictions of the angular size for an average-size galaxy (10 kpc).

In this specific case, even three curves overlap for small stances. Once reduced the scale, as shown in **Figure 8**, we can clearly underline the difference between these three curves.

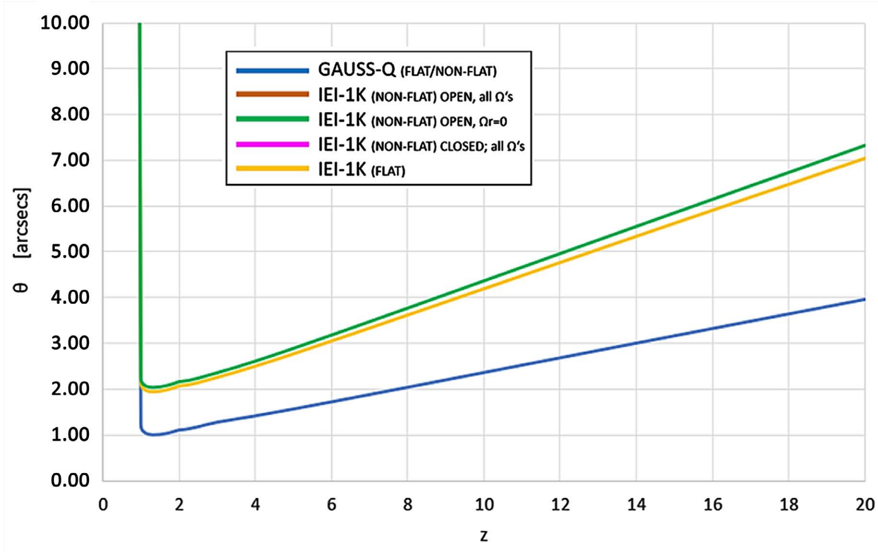

Figure 8. Predictions of the angular size for an average-size galaxy (10kpc). It is the plot of **Figure 7** with a smaller scale in order to highlight three curves at the bottom previously overlapping.

All calculations for the plots are resumed in the following **Tables 1-4**. Each table represents a specific cosmological scenario analyzed in the previous chapters.

Table 1. Computational methods applied to a non-flat (open) Universe. Gaussian quadrature vs incomplete elliptic integral of the first kind (int. 260.00) due to the quartic polynomial which admits four roots.

NON-FLAT (OPEN)	GAUSSIAN QUADRATURE					INCOMPLETE ELLIPTIC INTEGRAL OF THE FIRST KIND						
Ωr, Ωm, Ωk, ΩΛ	[Mpc]	[Gyrs]	[Gyrs]	[Gyrs]	[arcsec]	[rad]	[m]	[m]	[Gyrs]	[Gyrs]	[Gyrs]	[arcsec]
z	dtc	dtc	dL	dA	θ	arccos( )	F[r1,z]	dtc	dtc	dL	dA	θ
0	0	0	0	0	div	1.583	2.815	0	0	0	0	div
1	3402.490	11.105	22.210	5.553	1.212	1.762	3.455	1.494E+24	0.158	0.316	0.079	85.179
2	5313.890	17.343	52.030	5.781	1.164	1.890	3.816	2.336E+24	0.247	0.741	0.082	81.728
3	6508.430	21.242	84.969	5.311	1.267	1.987	4.040	2.859E+24	0.302	1.209	0.076	89.052
4	7335.350	23.941	119.705	4.788	1.405	2.065	4.198	3.226E+24	0.341	1.705	0.068	98.633
5	7949.240	25.945	155.668	4.324	1.556	2.128	4.312	3.494E+24	0.369	2.216	0.062	109.292
6	8427.640	27.506	192.543	3.929	1.712	2.182	4.403	3.707E+24	0.392	2.743	0.056	120.185
7	8813.820	28.767	230.132	3.596	1.871	2.227	4.475	3.874E+24	0.410	3.276	0.051	131.422
8	9133.950	29.811	268.302	3.312	2.031	2.267	4.536	4.016E+24	0.425	3.821	0.047	142.626
9	9404.900	30.696	306.957	3.070	2.192	2.302	4.587	4.136E+24	0.437	4.372	0.044	153.886
10	9638.070	31.457	346.024	2.860	2.353	2.333	4.631	4.238E+24	0.448	4.928	0.041	165.182
11	9841.480	32.121	385.447	2.677	2.513	2.361	4.670	4.329E+24	0.458	5.491	0.038	176.439
12	10020.960	32.706	425.183	2.516	2.674	2.386	4.704	4.407E+24	0.466	6.056	0.036	187.731
13	10180.860	33.228	465.196	2.373	2.834	2.409	4.734	4.478E+24	0.473	6.627	0.034	198.969
14	10324.480	33.697	505.455	2.246	2.995	2.430	4.761	4.542E+24	0.480	7.201	0.032	210.190
15	10454.420	34.121	545.938	2.133	3.155	2.449	4.786	4.599E+24	0.486	7.777	0.030	221.438

Continued

16	10572.720	34.507	586.623	2.030	3.314	2.466	4.807	4.649E+24	0.491	8.354	0.029	232.738
17	10681.000	34.861	627.491	1.937	3.474	2.483	4.828	4.698E+24	0.497	8.939	0.028	243.833
18	10780.620	35.186	668.530	1.852	3.633	2.498	4.847	4.742E+24	0.501	9.523	0.026	255.036
19	10872.660	35.486	709.723	1.774	3.792	2.512	4.864	4.782E+24	0.505	10.108	0.025	266.218
20	10958.030	35.765	751.061	1.703	3.950	2.526	4.881	4.821E+24	0.510	10.701	0.024	277.242

Table 2. Computational methods applied to a non-flat (open) Universe without radiation contribution. Gaussian quadrature vs incomplete elliptic integral of the first kind (int. 239.00) due to the cubic polynomial which admits three roots.

NON-FLAT (OPEN)	GAUSSIAN QUADRATURE					INCOMPLETE ELLIPTIC INTEGRAL OF THE FIRST KIND						
Ωm, Ωk, ΩΛ	[Mpc]	[Gyrs]	[Gyrs]	[Gyrs]	[arcsec]	[rad]	[m]	[m]	[Gyrs]	[Gyrs]	[Gyrs]	[arcsec]
z	dtc	dtc	dL	dA	θ	arccos( )	F[r1,z]	dtc	dtc	dL	dA	θ
0	0.000	0.000	0.000	0.000	div	1.582	2.811	0	0	0	0	div
1	3402.820	11.106	22.212	5.553	1.211	1.762	3.455	5.683E+25	6.007	12.014	3.003	2.240
2	5314.790	17.346	52.039	5.782	1.163	1.890	3.816	8.867E+25	9.372	28.116	3.124	2.153
3	6509.900	21.247	84.988	5.312	1.267	1.987	4.040	1.084E+26	11.461	45.842	2.865	2.348
4	7337.350	23.948	119.738	4.790	1.405	2.065	4.198	1.223E+26	12.930	64.649	2.586	2.602
5	7951.720	25.953	155.717	4.325	1.555	2.129	4.314	1.326E+26	14.016	84.093	2.336	2.880
6	8430.570	27.516	192.610	3.931	1.711	2.182	4.403	1.405E+26	14.851	103.954	2.122	3.171
7	8817.160	28.777	230.219	3.597	1.870	2.228	4.477	1.470E+26	15.534	124.275	1.942	3.465
8	9137.690	29.824	268.412	3.314	2.030	2.268	4.537	1.523E+26	16.101	144.906	1.789	3.761
9	9409.010	30.709	307.091	3.071	2.191	2.303	4.589	1.568E+26	16.578	165.784	1.658	4.058
10	9642.530	31.471	346.184	2.861	2.351	2.334	4.633	1.607E+26	16.988	186.868	1.544	4.356
11	9846.290	32.136	385.636	2.678	2.512	2.362	4.671	1.641E+26	17.348	208.178	1.446	4.653
12	10026.290	32.724	425.409	2.517	2.673	2.387	4.705	1.671E+26	17.663	229.614	1.359	4.952
13	10186.300	33.246	465.444	2.375	2.833	2.410	4.735	1.698E+26	17.946	251.248	1.282	5.248
14	10330.230	33.716	505.737	2.248	2.993	2.431	4.763	1.722E+26	18.201	273.015	1.213	5.544
15	10460.470	34.141	546.254	2.134	3.153	2.450	4.787	1.743E+26	18.428	294.844	1.152	5.841
16	10579.040	34.528	586.973	2.031	3.312	2.468	4.810	1.763E+26	18.640	316.872	1.096	6.136
17	10687.600	34.882	627.879	1.938	3.472	2.484	4.829	1.781E+26	18.825	338.854	1.046	6.433
18	10787.490	35.208	668.956	1.853	3.630	2.500	4.849	1.798E+26	19.009	361.172	1.000	6.724
19	10879.790	35.509	710.189	1.775	3.789	2.514	4.866	1.813E+26	19.169	383.373	0.958	7.019
20	10965.420	35.789	751.567	1.704	3.948	2.527	4.882	1.827E+26	19.315	405.618	0.920	7.314

Table 3. Computational methods applied to a non-flat (closed) Universe. Gaussian quadrature vs incomplete elliptic integral of the first kind (int. 260.00) due to the quartic polynomial which admits four roots.

NON-FLAT (CLOSED)	GAUSSIAN QUADRATURE					INCOMPLETE ELLIPTIC INTEGRAL OF THE FIRST KIND						
Ωr, Ωm, Ωk, ΩΛ	[Mpc]	[Gyrs]	[Gyrs]	[Gyrs]	[arcsec]	[rad]	[m]	[m]	[Gyrs]	[Gyrs]	[Gyrs]	[arcsec]
z	dtc	dtc	dL	dA	θ	arccos( )	F[r1,z]	dtc	dtc	dL	dA	θ
0	0.000	0.000	0.000	0.000	div	1.582	2.811	0	0	0	0	div
1	3403.760	11.109	22.218	5.555	1.211	1.761	3.452	1.003E+24	0.106	0.212	0.053	126.957
2	5315.070	17.347	52.042	5.782	1.163	1.889	3.814	1.568E+24	0.166	0.497	0.055	121.777

Continued

3	6508.500	21.242	84.970	5.311	1.267	1.987	4.040	1.922E+24	0.203	0.813	0.051	132.453
4	7334.040	23.937	119.684	4.787	1.405	2.064	4.196	2.165E+24	0.229	1.144	0.046	146.958
5	7946.540	25.936	155.615	4.323	1.556	2.128	4.312	2.348E+24	0.248	1.489	0.041	162.650
6	8423.620	27.493	192.451	3.928	1.713	2.181	4.402	2.488E+24	0.263	1.841	0.038	179.069
7	8808.570	28.749	229.995	3.594	1.872	2.227	4.475	2.603E+24	0.275	2.201	0.034	195.635
8	9127.580	29.791	268.115	3.310	2.032	2.267	4.536	2.698E+24	0.285	2.566	0.032	212.331
9	9397.490	30.672	306.715	3.067	2.193	2.302	4.587	2.778E+24	0.294	2.936	0.029	229.109
10	9629.720	31.429	345.724	2.857	2.355	2.333	4.631	2.847E+24	0.301	3.310	0.027	245.940
11	9832.260	32.091	385.086	2.674	2.516	2.361	4.670	2.907E+24	0.307	3.688	0.026	262.712
12	10010.940	32.674	424.758	2.513	2.677	2.386	4.704	2.960E+24	0.313	4.067	0.024	279.536
13	10170.090	33.193	464.704	2.371	2.837	2.408	4.733	3.005E+24	0.318	4.447	0.023	296.479
14	10313.020	33.660	504.894	2.244	2.998	2.429	4.760	3.048E+24	0.322	4.833	0.021	313.206
15	10442.320	34.082	545.306	2.130	3.158	2.448	4.784	3.086E+24	0.326	5.219	0.020	329.972
16	10560.000	34.466	585.917	2.027	3.318	2.466	4.807	3.122E+24	0.330	5.610	0.019	346.590
17	10667.720	34.817	626.711	1.934	3.478	2.482	4.827	3.153E+24	0.333	5.999	0.019	363.355
18	10766.810	35.141	667.673	1.850	3.637	2.498	4.847	3.184E+24	0.337	6.394	0.018	379.810
19	10858.340	35.439	708.789	1.772	3.797	2.512	4.864	3.211E+24	0.339	6.787	0.017	396.470
20	10943.240	35.717	750.047	1.701	3.956	2.525	4.880	3.235E+24	0.342	7.181	0.016	413.133

Table 4. Computational methods applied to flat Universe. Gaussian quadrature vs incomplete elliptic integral of the first kind (int. 239.00) due to the cubic polynomial which admits three roots.

FLAT	GAUSSIAN QUADRATURE					INCOMPLETE ELLIPTIC INTEGRAL OF THE FIRST KIND						
Ωm, ΩΛ	[Mpc]	[Gyrs]	[Gyrs]	[Gyrs]	[arcsec]	[rad]	[m]	[m]	[Gyrs]	[Gyrs]	[Gyrs]	[arcsec]
z	dtc	dtc	dL	dA	θ	arccos( )	F[r1,z]	dtc	dtc	dL	dA	θ
0	0.000	0.000	0.000	0.000	div	1.582	2.811	0	0	0	0	div
1	3403.120	11.107	22.214	5.554	1.211	1.762	3.455	5.900E+25	6.236	12.473	3.118	2.158
2	5315.080	17.347	52.042	5.782	1.163	1.890	3.816	9.205E+25	9.730	29.189	3.243	2.074
3	6509.920	21.247	84.988	5.312	1.267	1.987	4.040	1.126E+26	11.897	47.590	2.974	2.262
4	7337.030	23.947	119.733	4.789	1.405	2.065	4.198	1.270E+26	13.422	67.112	2.684	2.506
5	7951.080	25.951	155.704	4.325	1.555	2.129	4.314	1.376E+26	14.549	87.296	2.425	2.774
6	8429.610	27.513	192.588	3.930	1.712	2.182	4.403	1.458E+26	15.416	107.911	2.202	3.055
7	8815.910	28.773	230.187	3.597	1.870	2.228	4.477	1.526E+26	16.126	129.005	2.016	3.338
8	9136.160	29.819	268.367	3.313	2.031	2.268	4.537	1.581E+26	16.713	150.419	1.857	3.623
9	9407.240	30.703	307.033	3.070	2.191	2.303	4.589	1.628E+26	17.209	172.090	1.721	3.909
10	9640.540	31.465	346.112	2.860	2.352	2.334	4.633	1.668E+26	17.634	193.974	1.603	4.197
11	9844.080	32.129	385.549	2.677	2.513	2.362	4.671	1.704E+26	18.008	216.093	1.501	4.483
12	10023.700	32.715	425.299	2.517	2.673	2.387	4.705	1.735E+26	18.334	238.343	1.410	4.770
13	10183.730	33.238	465.327	2.374	2.834	2.410	4.735	1.762E+26	18.628	260.798	1.331	5.056
14	10327.490	33.707	505.603	2.247	2.994	2.431	4.763	1.787E+26	18.893	283.391	1.260	5.341
15	10457.570	34.131	546.102	2.133	3.154	2.450	4.787	1.810E+26	19.128	306.048	1.195	5.627

Continued

16	10576.000	34.518	586.805	2.030	3.313	2.468	4.810	1.830E+26	19.348	328.912	1.138	5.911
17	10684.430	34.872	627.693	1.937	3.473	2.484	4.829	1.849E+26	19.540	351.728	1.086	6.197
18	10784.180	35.197	668.750	1.852	3.632	2.500	4.849	1.867E+26	19.731	374.893	1.038	6.478
19	10876.360	35.498	709.965	1.775	3.790	2.514	4.866	1.882E+26	19.897	397.935	0.995	6.762
20	10961.880	35.777	751.325	1.704	3.949	2.527	4.882	1.897E+26	20.049	421.024	0.955	7.047

4. Conclusions

Compared to the different distance scales and magnitude orders that we obtain by means of the incomplete elliptic integrals, we can state that the predictions of Gaussian quadrature both for a non-flat and for a flat Universe are basically identical. The variation of its values is enclosed in a closer scale in the same magnitude order. For this reason, we approximate the Gaussian quadrature prediction with a single curve (the blue one) in each plot in the framework of this inquiry.

When we discuss the distances in cosmology, we can definitively state that it is not possible to solely perform an analytical calculation. Despite many efforts to provide only an analytical solution, this has to be followed by a numerical one in order to account for the redshift, the integration variable, in the integrals. In this context and based on the outcome of this inquiry, the predictions of the incomplete elliptic integrals of the first kind would drastically change the argumentations in cosmology as we have a big deviation in value depending on the type of Universe that we are assuming through the assumption of the existence or absence of specific cosmological parameters.

Going into detail, concerning the transverse comoving distance, a flat Universe or a non-flat Universe without radiation approached through the incomplete elliptic integrals have the closer curve, and therefore prediction, to that of the Gaussian quadrature. The deviation appears to stabilize among 15 Glyrs difference for increasing redshift at least within z = 20. Down to the last curve, a non-flat closed Universe would differentiate the most from the Gaussian quadrature method. Any value below the Gaussian quadrature prediction basically means that we calculate and predict closer distance in space. Exactly the same observations can be done for the luminosity distance and the angular diameter distance. In these plots, all curves appear to follow the same trend and deviations to each other, except for the modulus of the deviation. Even with the angular size plot, we can observe a consistent behavior of the curves as the latter are now turned upside down due to the inverse proportionality to the transverse comoving distance. The predictions of the angular size according to the incomplete elliptical integrals of the first kind would worsen the cosmological predictions as we would expect to observe bigger galaxies for increasing redshift. We assumed an average-size galaxy equal to 10kpc without arguing about the evolutionary stage and therefore the expected size of the galaxies at higher redshifts.

Remaining on the topic of the elliptic integrals, when we include all omega density parameter in the calculation, all various distances calculated (transverse comoving, luminosity and angular diameter) have smaller values compared to the Gaussian quadrature. It translates into bigger values for the angular size of the galaxies. A closed Universe shows the smallest distances even compared to an open one. At the time that we assume to neglect one or more omega density parameters in the equations, the distances increase as shown with a flat Universe (without curvature and radiation – orange curve) as well as with a non-flat Universe (without radiation – green curve). This is because by neglecting existing parameters for which we are assuming important physical meanings, we are basically removing the constraints and the correlations between physics, mathematics and the reality of the Universe that surrounds us. By removing one by one omega density parameters, we end up with bigger distances, despite being smaller than the Gaussian quadrature ones, as we are releasing the Universe from the physical and mathematical resistance exerted by the parameters that we removed. With this logic, it is important to stress that the calculation of the distances in cosmology should be performed without neglecting parameters but rather making efforts to include them all and by providing more exact values based on observational data. For instance, by neglecting the radiation from the equations we change the reality of our Universe in which we do have the radiation and it is furthermore the only tool we have to measure distances through the spectrum of the astronomical sources. It is also the only way to measure the redshift and accordingly the only way to compare measurements with predictions. Ultimately, we can state that the removal of the radiation, in the form of the omega density parameter, makes scientifically no sense.

Indeed, we should include all parameters defined by Friedmann in General Relativity and we have therefore to observe their outcomes in terms of predictions from the equations to then make a comparison with observational data. Despite some parameters can be mathematically approximated to zero, their influence on a complex integral, such as that of the comoving distance, cannot be neglected. From the mathematical standpoint, in the main integral, all omega density parameters appear to multiply the redshift in different power orders. This translates into the fact that, for instance, a tiny omega density radiation can still influence the integral if multiplied by the redshift in a quartic polynomial where the fourth degree belongs exactly to the radiation term. It is a fact that we have to consider all omega density parameters without approximation as any parameter affects the calculation independently of the computational method.

With regard to the difference between the curve of the transverse comoving distance in a non-flat cosmology through an incomplete elliptical integral or the solution by means of the Gaussian quadrature, it can indeed open different scenarios.

a) The curves calculated by the incomplete elliptical integral of the first kind reflect the effective behavior of the Universe. In this case, we are currently overestimating the distance values in cosmology due to the Gaussian quadrature method. This remark has a consequence on the whole cosmology as, for instance, the study of the distance modulus of the supernovae Ia might require a re-investigation. The same can be stated with the Hubble tension and the influence that this decisive parameter has on the integrals of the transverse comoving distance, in which it is inversely proportional;

b) The curves calculated by the incomplete elliptical integral of the first kind evolve differently, in defect, from the Gaussian quadrature-curve and the reason might be attributed to the analytical solution obtained by Legendre-Jacobi's approach discussed in Byrd-Friedmann's handbook of elliptic integrals. Alternatively, this class of solutions might also be intrinsically an approximation compared to the Gaussian quadrature. A deeper mathematical inquiry concerning the correctness of the approach might follow this cosmological study for a better understanding of the mathematical approach and the comparison between the two methods.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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