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Research Article

On Annihilated Points and Approximate Fixed Points of General Higher-Order Nonexpansive Mappings

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In this paper, we extend the results obtained by Ezearn on annihilated points for his higher-order nonexpansive mappings to the context of general higher-order nonexpansive mappings. Precisely in his thesis, Ezearn introduced the concept of annihilated points, which extends the notion of fixed points, and it is only meaningful in the context of higher-order nonexpansive mappings and gave some mild conditions when the annihilated points could exist in strictly convex Banach spaces. In the last direction, we also extend Ezearn's result on the approximate fixed point sequence for higher-order nonexpansive mappings to general higher-order nonexpansive mappings.

1. Introduction

Given a complete metric space $(\mathcal{X}, d)$, the most well-studied examples of such mappings are those that can be immediately put in the form

$$d(Tx, Ty) \le c \cdot d(x, y), \tag{1}$$

For all $x, y \in \mathcal{X}$ where $c > 0$ is a fixed real number. Such mappings are referred to as *Lipschitz* continuous mappings. Lipschitz continuous mappings are generally classified into three categories: T is a

(i) *contraction* mapping if $0 < c < 1$

(ii) *nonexpansive* mapping if $c = 1$

(iii) *expansive* mapping if $c > 1$

In [1], the concept of *mean nonexpansive mappings* was introduced which is often seen as a generalization of nonexpansive mappings. Thus, let $\mathcal{C}$ be a nonempty subset of a Banach space $\mathcal{X}$, and let T be a self-mapping on $\mathcal{C}$. Then T is called a *mean nonexpansive* (or *α-nonexpansive*) if

$$\sum_{k=1}^{n} \alpha_k \left\| T^k x - T^k y \right\| \le \|x - y\|, \tag{2}$$

For all $x, y \in \mathcal{C}$ and for some $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, we have $\sum_{k=1}^{n} \alpha_k = 1$, $\alpha_k \ge 0$ for all k, and $\alpha_1, \alpha_n > 0$.

Clearly, it is seen that all nonexpansive mappings are mean nonexpansive mappings, but the reverse is not always true, as demonstrated in ([2], Examples 2.3 and 2.4). A more general class of (α, p)-nonexpansive maps was further introduced in [1]. That is, a self-map T on a subset $\mathcal{C}$ of a Banach space $\mathcal{X}$ is called (α, p)-*nonexpansive* if

$$\sum_{k=1}^{n} \alpha_k \left\| T^k x - T^k y \right\|^p \le \|x - y\|^p, \tag{3}$$

For all $x, y \in \mathscr{C}$ and for some $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, we have $\sum_{k=1}^{n} \alpha_k = 1$, $\alpha_k \geq 0$ for all k, $\alpha_1, \alpha_n > 0$ and for some $p \in [1, \infty)$. It is obvious that (α, p)-nonexpansive map for $p > 1$ is also α-nonexpansive, but the reverse is not always true, as shown in [3].

Now, given a metric space $(\mathscr{X}, d)$, a more general class of mappings which extend inequality (1) can be put in the following form:

$$d(T^r x, T^r y) \leq \sum_{k=0}^{r-1} c_k d\left(T^k x, T^k y\right) \quad \forall x, y \in \mathscr{X}, \tag{4}$$

where $r \in \mathbb{N}$ and $c_k \geq 0$, for all $0 \leq k \leq r-1$. Such mappings are called *higher-order Lipschitz mappings* (or r*th-order Lipschitz* mappings, for short) which was introduced by Ezearn [4] in 2015.

Now, to every higher-order Lipschitz mapping, Ezearn associated a polynomial which is defined as

$$p(z) = z^r - \sum_{k=0}^{r-1} c_k z^k, \tag{5}$$

and for *r th-order nonexpansive mapping*, we have $p(1) = 0$.

Ezearn [5] in his thesis introduced the concept of *annihilated points* of a higher-order nonexpansive mapping as defined below:

Definition 1 (Annihilated point of T). Let $T : \mathscr{S} \longrightarrow \mathscr{X}$ be a higher-order nonexpansive mapping on a subset $\mathscr{S}$ of a Banach space $\mathscr{X}$, and let p be the associated polynomial of T. Then x is an annihilated point (respectively, a totally annihilated point) of T if $p(T)$ annihilates x (respectively, the Picard iterates of x) that is, $p(T)x = 0$ (respectively, $p(T)T^n x = 0$ for all $n \geq 0$).

Ezearn is denoted by $A(T)$ (respectively, $A_\infty(T)$) the set of annihilated (respectively, a totally annihilated) points of T. Ezearn, in an attempt to prove a fixed point result for higher-order nonexpansive mappings, proved the following theorems on sufficient conditions for an annihilated point when the Banach space is strictly convex: a *strictly convex* Banach space is a Banach space such that whenever $x \neq 0$ and $y \neq 0$, then $\|x + y\| = \|x\| + \|y\|$ if and only if $x = ky$ for some constant $k > 0$.

Theorem 2. *Let $\mathscr{C}$ be a convex subset of a strictly Banach space $\mathscr{X}$, and let $T : \mathscr{C} \longrightarrow \mathscr{C}$ an rth-order nonexpansive mapping of the form*

$$\|T^r y - T^r x\| \leq \left\| \sum_{k=0}^{r-1} c_k \left(T^k y - T^k x\right) \right\|. \tag{6}$$

Suppose $u, v \in A(T)$ and $T^r x - p(T)x \in conv\{T^r, T^r v\}$. Then, $x \in A(T)$.

Theorem 3. *Let T be an rth-order nonexpansive mapping on a convex subset $\mathscr{C}$ of a strictly convex Banach space $\mathscr{X}$. Suppose $u, v \in Fix(T)$ and $\{T^k x\}_{k=0}^{r-1} \subset conv\{u, v\}$. Then $x \in A_\infty(T)$.*

With a mild condition on the set of totally annihilated points, $A_\infty(T)$, Ezearn proved the following fixed point result in a general Banach space.

Theorem 4. *Let T be an affine higher-order nonexpansive mapping on a convex subset $\mathscr{C}$ of a Banach space $\mathscr{X}$. Then, $Fix(T) = \varnothing$ only if $A_\infty(T) = \varnothing$. In particular, the identity $A(T) = A_\infty(T)$ holds, and if $x \in A_\infty(T)$, then*

$$\frac{1}{\sum_{k=0}^{r-1} b_k} \sum_{k=0}^{r-1} b_k T^k x \in Fix(T). \tag{7}$$

Finally, Ezearn proved the following approximate fixed point sequence result for his higher-order nonexpansive mapping in a general Banach space.

Theorem 5. *Let $\mathscr{C}$ be a closed bounded star-convex subset of a Banach space, and let T be an affine rth-order nonexpansive self-mapping on $\mathscr{C}$. Then, T has an approximate fixed point sequence in $\mathscr{C}$. That is, there exists $\{x_n\}_{n\geq 1} \subset \mathscr{C}$ such that $\lim_{n\longrightarrow\infty}(x_n - Tx_n) = 0$.*

In 2021, the author [6] introduced the following mappings which generalize both inequality (3) and (4).

Definition 6 (General higher-order Lipschitz mappings). Given a metric space $(\mathscr{X}, d)$, a self-map T on $\mathscr{X}$ is called a (r, p)-general rth-order Lipschitz mapping if

$$\sum_{k=l+1}^{r} \alpha_k d\left(T^k x, T^k y\right)^p \leq \sum_{k=0}^{l} \alpha_k d\left(T^k x, T^k y\right)^p \quad \forall x, y \in \mathscr{X}, \tag{8}$$

where $p \geq 1$, $r \in \mathbb{N}$, and $\alpha_k \geq 0$ for all k, $\alpha_0 \cdot \alpha_r \neq 0$, and $l \in \{0, \cdots, r-1\}$.

It is obvious that inequality (8) reduces to (3) when $l = 0$. In the same vein, inequality (8) reduces to (4) when $p = 1$ and $l = r - 1$.

Now to every (r, p)-general higher-order Lipschitz mapping, the author associated the following polynomial:

$$h(z) = \sum_{k=l+1}^{r} \alpha_k z^k - \sum_{k=0}^{l} \alpha_k z^k. \tag{9}$$

The author classified (r, p)-general higher-order Lipschitz mappings as follows:

(i) T is (r, p)-general higher-order contraction mapping if $h(1) > 0$

(ii) T is (r, p)-general higher-order non-expansive mapping if $h(1) = 0$

(iii) T is (r,p)-general higher-order expansive mapping if $h(1) < 0$

In this paper, we generalize Theorem 2 and Theorem 3 to (r,p)-general higher-order nonexpansive mapping when $p = 1$ except that in the second case (Theorem 3), it will not be totally annihilated points but just annihilated points because in Ezearn's case, all the constants are on the right, and therefore, by induction, he could obtain that result for a totally annihilated point. In the other direction, we generalize Theorem 4 to (r,p)-general higher-order nonexpansive mappings, but in the context of an *affine subset* of a given Banach space. In the last direction, we generalize Theorem 5 to (r,p)-general higher-order nonexpansive mappings. That is, in this paper, we prove the following results:

Theorem 7. *Let $\mathscr{C}$ be a convex subset of a strictly convex Banach space $\mathscr{X}$, and define $T : \mathscr{C} \longrightarrow \mathscr{C}$ to be a (r,1)-general-higher order nonexpansive mapping of the form*

$$\left\| \sum_{k=l+1}^{r} \alpha_k \left(T^k y - T^k x\right) \right\| \leq \left\| \sum_{k=0}^{l} \alpha_k \left(T^k y - T^k x\right) \right\|. \tag{10}$$

Suppose $u, v \in A(T)$ and $\sum_{k=l+1}^{r} \alpha_k T^k x - h(T)x \in conv \{\sum_{k=l+1}^{r} \alpha_k T^k u, \sum_{k=l+1}^{r} \alpha_k T^k v\}$. Then, $x \in A(T)$.

Theorem 8. *Let T be an (r,1)-general higher-order nonexpansive mapping on a convex subset $\mathscr{C}$ of a strictly convex Banach space $(\mathscr{X}, \|\cdot\|)$. Suppose $u, v \in Fix(T)$ and $\{T^k x\}_{k=0}^{r-1} \subset conv\{u, v\}$. Then, $x \in A(T)$.*

Theorem 9. *Let T be an affine general higher-order nonexpansive mapping on an affine subset $\mathscr{C}$ of a Banach space $\mathscr{X}$. The $Fix(T) = \varnothing$ only if $A_\infty(T) = \varnothing$. In particular, the identity $A(T) = A_\infty(T)$ holds and if $x \in A_\infty(T)$, then*

$$\frac{1}{\sum_{k=0}^{r-1} b_k} \sum_{k=0}^{r-1} b_k T^k x \in Fix(T). \tag{11}$$

Theorem 10. *Let $\mathscr{C}$ be a closed, bounded star-convex subset of a Banach space, and let T be an affine (r,p)-general higher-order nonexpansive self-mapping on $\mathscr{C}$. Then T has an approximate fixed point sequence in $\mathscr{C}$. That is, there exists $\{x_n\}_{n\geq 1} \subset \mathscr{C}$ such that $\lim_{n\longrightarrow\infty}(x_n - Tx_n) = 0$.*

From Definition 1, for any *(r,p)-general higher-order nonexpansive mapping*, the fixed point set is always a subset of the annihilated point set and they *coincide* when $l = 0$ and $r = 1$. To see this, for $l = 0$ and $r = 1$, we have the following:

$$h(T) = \sum_{k=l+1}^{r} \alpha_k T^k - \sum_{k=0}^{l} \alpha_k T^k = \alpha_1 T - \alpha_0. \tag{12}$$

Given that $T(z) = z$, then we have

$$h(T)z = \alpha_1 Tz - \alpha_0 z = \alpha_1 z - \alpha_0 z = (\alpha_1 - \alpha_0)z = 0 \cdot z = 0. \tag{13}$$

In the same vein, since for any *(r,p)-general higher-order nonexpansive mapping*, we have

$$\sum_{k=l+1}^{r} \alpha_k = \sum_{k=0}^{l} \alpha_k, \tag{14}$$

then, we have the following:

$$h(T)z = \sum_{k=l+1}^{r} \alpha_k T^k z - \sum_{k=0}^{l} \alpha_k T^k z. \tag{15}$$

Now, since $Tz = z \Rightarrow T^k z = z$ for $k \geq 1$, then the above equation reduces to

$$h(T)z = \sum_{k=l+1}^{r} \alpha_k z - \sum_{k=0}^{l} \alpha_k z, = \left(\sum_{k=l+1}^{r} \alpha_k - \sum_{k=0}^{l} \alpha_k\right) \cdot z, = 0 \cdot z, = 0. \tag{16}$$

2. Preliminaries

Proposition 11. *Define T to be an (r,p)-general higher-order Lipschitz mapping, and let $h(z)$ be the associated polynomial for T as stated in Definition 6.*

(i) If $h(1) > 0$, then we can always find a certain $\lambda \in (0, 1)$, which is unique and positive if $\alpha_k \neq 0$, such that $h(\lambda) = 0$

(ii) If $h(1) = 0$, then there exists 1 as the only positive root of h

(iii) If $h(1) < 0$, then we can find a unique positive $\lambda > 1$ such that $h(\lambda) = 0$

Now, let us define T to be an *(r,p)-general higher-order Lipschitz mapping* on a complete metric space $(\mathscr{X}, d)$ as given in inequality (8) and let λ be the unique root of the polynomial $h(z)$ as guaranteed by Proposition 11. Define the following on the space $\mathscr{X}$:

$$D_p(x, y) = \left(\sum_{k=0}^{r-1} b_k d\left(T^k x, T^k y\right)^p\right)^{1/p}, \tag{17}$$

where

$$b_k = \sum_{j=0}^{k} \alpha_j \lambda^{j-k-1} \operatorname{sgn}(j), \tag{18}$$

and $p \geq 1$ for all $x, y \in \mathcal{X}$, $0 \leq k \leq r-1$ and

$$\operatorname{sgn}(j) = \begin{pmatrix} 1 & \text{if } j \leq l \\ -1 & \text{if } j > l \end{pmatrix}. \tag{19}$$

Corollary 12. *b_k stated in equation (17) is non-negative.*

Lemma 13. *$D_p(x, y)$ stated in equation (17) is a metric on the space $\mathcal{X}$.*

Proposition 14. *Define b_k in equation (17). Then the following results hold:*

$$\begin{gathered} \alpha_0 = \lambda b_0, \quad b_{r-1} - \alpha_r = 0, \\ \lambda b_k = b_{k-1} + \alpha_k \operatorname{sgn}(k), \quad 1 \leq k \leq r-1. \end{gathered} \tag{20}$$

Lemma 15. *Given a metric space $(\mathcal{X}, d)$ (not necessarily complete) and define $T : \mathcal{X} \longrightarrow \mathcal{X}$ to be an (r, p)-general higher-order Lipschitz mapping. Then*

$$D_p(Tx, Ty) \leq \lambda^{1/p} D_p(x, y). \tag{21}$$

Moreover, a sequence $\{x_n\}_{n\geq 1} \subset (\mathcal{X}, D_p)$ is Cauchy in $(\mathcal{X}, D_p)$ if and only if the sequence $\{T^k x_n\}_{n\geq 1} \subset (\mathcal{X}, d)$ is Cauchy in $(\mathcal{X}, d)$ for all $0 \leq k \leq r-1$.

Theorem 16. *Define the mapping,*

$$\bar{T} : \bar{\mathcal{X}} \longrightarrow \bar{\mathcal{X}}, \quad [x_n] \longrightarrow [Tx_n]. \tag{22}$$

Then, we have

$$\bar{D}_p\left(\bar{T}[y_n], \bar{T}[x_n]\right) \leq \lambda^{1/p} \bar{D}_p([y_n], [x_n]). \tag{23}$$

In particular, if $(\mathcal{X}, d)$ is complete, then T has a fixed point in $(\mathcal{X}, d)$ if and only if $\bar{T}$ has a fixed point in $(\bar{\mathcal{X}}, \bar{D}_p)$.

3. Main Result

We prove the main result of this paper, which is already stated in Theorem 7, Theorem 8, Theorem 9, and Theorem 10. The proofs follow similarly as in Ezearn [5] except for few modifications as necessary.

Proof of Theorem 17. Let

$$\sum_{k=l+1}^{r} \alpha_k T^k x - h(T)x = c \sum_{k=l+1}^{r} \alpha_k T^k u + (1-c) \sum_{k=l+1}^{r} \alpha_k T^k v, \tag{24}$$

for some $c \in [0, 1]$. Then, given that $\sum_{k=l+1}^{r} \alpha_k T^k - h(T) = \sum_{k=0}^{l} \alpha_k T^k$, then the following identity holds:

$$\begin{aligned} \sum_{k=0}^{l} \alpha_k \left(T^k x - T^k v\right) &= \sum_{k=0}^{l} \alpha_k T^k x - \sum_{k=0}^{l} \alpha_k T^k v, \\ &= \sum_{k=l+1}^{r} \alpha_k T^k x - h(T)x - \sum_{k=0}^{l} \alpha_k T^k v, \\ &= c \sum_{k=l+1}^{r} \alpha_k T^k u + (1-c) \sum_{k=l+1}^{r} \alpha_k T^k v \\ &\quad - \sum_{k=0}^{l} \alpha_k T^k v, = c \sum_{k=0}^{l} \alpha_k T^k u \\ &\quad + (1-c) \sum_{k=0}^{l} \alpha_k T^k v - \sum_{k=0}^{l} \alpha_k T^k v, \\ &= c \sum_{k=0}^{l} \alpha_k T^k u - c \sum_{k=0}^{l} \alpha_k T^k v, \\ &= c\left(\sum_{k=l+1}^{r} \alpha_k T^k u - h(T)u\right) \\ &\quad - c\left(\sum_{k=l+1}^{r} \alpha_k T^k v - h(T)v\right), \\ &= c \sum_{k=l+1}^{r} \alpha_k \left(T^k u - T^k v\right). \end{aligned} \tag{25}$$

Hence, we have

$$\sum_{k=0}^{l} \alpha_k \left(T^k x - T^k v\right) = c \sum_{k=l+1}^{r} \alpha_k \left(T^k u - T^k v\right). \tag{26}$$

Similarly, one can also have

$$\begin{aligned} \sum_{k=0}^{l} \alpha_k \left(T^k u - T^k x\right) &= \sum_{k=0}^{l} \alpha_k T^k u - \sum_{k=0}^{l} \alpha_k T^k x, \\ &= \sum_{k=0}^{l} \alpha_k T^k u - \left(\sum_{k=l+1}^{r} \alpha_k T^k x - h(T)x\right), \\ &= \sum_{k=0}^{l} \alpha_k T^k u - \left(c \sum_{k=l+1}^{r} \alpha_k T^k u + (1-c) \sum_{k=l+1}^{r} \alpha_k T^k v\right), \\ &= \sum_{k=0}^{l} \alpha_k T^k u - \left(c \sum_{k=0}^{l} \alpha_k T^k u + (1-c) \sum_{k=0}^{l} \alpha_k T^k v\right), \\ &= (1-c) \sum_{k=0}^{l} \alpha_k T^k u - (1-c) \sum_{k=0}^{l} \alpha_k T^k v, \\ &= (1-c)\left(\sum_{k=l+1}^{r} \alpha_k T^k u - h(T)u\right) \\ &\quad - (1-c)\left(\sum_{k=l+1}^{r} \alpha_k T^k v - h(T)v\right). \end{aligned} \tag{27}$$

Hence, we have

$$\sum_{k=0}^{l} \alpha_k \left(T^k u - T^k x\right) = (1-c) \sum_{k=l+1}^{r} \alpha_k \left(T^k u - T^k v\right). \quad (28)$$

From equation (26), it follows that when $c = 0$, then

$$\left\| \sum_{k=l+1}^{r} \alpha_k \left(T^k x - T^k v\right) \right\| \le \left\| \sum_{k=0}^{l} \alpha_k \left(T^k x - T^k v\right) \right\| = 0. \quad (29)$$

and that implies that

$$\sum_{k=l+1}^{r} \alpha_k T^k x = \sum_{k=l+1}^{r} \alpha_k T^k v. \quad (30)$$

Note also that when $c = 0$, then $\sum_{k=l+1}^{r} \alpha_k T^k x - h(T) x = 0 \cdot \sum_{k=l+1}^{r} \alpha_k T^k u + (1-0) \sum_{k=l+1}^{r} \alpha_k T^k v$ and it follows that

$$\sum_{k=l+1}^{r} \alpha_k T^k x - h(T) x = \sum_{k=l+1}^{r} \alpha_k T^k v. \quad (31)$$

Combining equation (30) and equation (31), we have that

$$\sum_{k=l+1}^{r} \alpha_k T^k x = \sum_{k=l+1}^{r} \alpha_k T^k v \coloneqq \sum_{k=l+1}^{r} \alpha_k T^k x - h(T) x, \quad (32)$$

giving $h(T) = 0$ or equivalently $x \in A(T)$.

Similarly, from equation (28), when $c = 1$, then

$$\left\| \sum_{k=l+1}^{r} \alpha_k \left(T^k u - T^k x\right) \right\| \le \left\| \sum_{k=0}^{l} \alpha_k \left(T^k u - T^k x\right) \right\| = 0. \quad (33)$$

and that implies that

$$\sum_{k=l+1}^{r} \alpha_k T^k u = \sum_{k=l+1}^{r} \alpha_k T^k x. \quad (34)$$

Note also that when $c = 1$, then $\sum_{k=l+1}^{r} \alpha_k T^k x - h(T) x = 1 \cdot \sum_{k=l+1}^{r} \alpha_k T^k u + (1-1) \sum_{k=l+1}^{r} \alpha_k T^k v$ and it follows that

$$\sum_{k=l+1}^{r} \alpha_k T^k x - h(T) x = \sum_{k=l+1}^{r} \alpha_k T^k u. \quad (35)$$

Combining equation (34) and equation (35), we have that

$$\sum_{k=l+1}^{r} \alpha_k T^k x = \sum_{k=l+1}^{r} \alpha_k T^k u \coloneqq \sum_{k=l+1}^{r} \alpha_k T^k x - h(T) x, \quad (36)$$

giving $h(T) = 0$ or equivalently $x \in A(T)$.

Hence, we assume that $c \in (0, 1)$. We observe that

$$\sum_{k=l+1}^{r} \alpha_k T^k u \ne \sum_{k=l+1}^{r} \alpha_k T^k x \ne \sum_{k=l+1}^{r} \alpha_k T^k v. \quad (37)$$

To see this, we note that if $\sum_{k=l+1}^{r} \alpha_k T^k u = \sum_{k=l+1}^{r} \alpha_k T^k x$, then we have the following

$$\begin{aligned} \left\| \sum_{k=0}^{l} \alpha_k \left(T^k x - T^k v\right) \right\| &= c \left\| \sum_{k=l+1}^{r} \alpha_k \left(T^k u - T^k v\right) \right\|, \\ &= c \left\| \sum_{k=l+1}^{r} \alpha_k \left(T^k x - T^k v\right) \right\|, \\ &\le c \left\| \sum_{k=0}^{l} \alpha_k \left(T^k x - T^k v\right) \right\|, \end{aligned} \quad (38)$$

leading to the contradiction that $c \ge 1$.

Also if $\sum_{k=l+1}^{r} \alpha_k T^k v = \sum_{k=l+1}^{r} \alpha_k T^k x$, then we have

$$\begin{aligned} \left\| \sum_{k=0}^{l} \alpha_k \left(T^k u - T^k x\right) \right\| &= (1-c) \left\| \sum_{k=l+1}^{r} \alpha_k \left(T^k u - T^k v\right) \right\|, \\ &= (1-c) \left\| \sum_{k=l+1}^{r} \alpha_k \left(T^k u - T^k x\right) \right\|, \\ &\le (1-c) \left\| \sum_{k=0}^{l} \alpha_k \left(T^k u - T^k x\right) \right\|, \end{aligned} \quad (39)$$

leading to the contradiction that $c \le 1$.

Now, given that

$$\begin{aligned} \left\| \sum_{k=l+1}^{r} \alpha_k \left(T^k u - T^k v\right) \right\| &= \left\| \sum_{k=l+1}^{r} \alpha_k \left(T^k u - T^k x + T^k x - T^k v\right) \right\|, \\ &\le \left\| \sum_{k=l+1}^{r} \alpha_k \left(T^k u - T^k x\right) \right\| \\ &\quad + \left\| \sum_{k=l+1}^{r} \alpha_k \left(T^k x - T^k v\right) \right\|, \\ &\le \left\| \sum_{k=0}^{l} \alpha_k \left(T^k u - T^k x\right) \right\| \\ &\quad + \left\| \sum_{k=0}^{l} \alpha_k \left(T^k x - T^k v\right) \right\|, \\ &= (1-c) \left\| \sum_{k=l+1}^{r} \alpha_k \left(T^k u - T^k v\right) \right\| \\ &\quad + c \left\| \sum_{k=l+1}^{r} \alpha_k \left(T^k u - T^k v\right) \right\|, \\ &= \left\| \sum_{k=l+1}^{r} \alpha_k \left(T^k u - T^k v\right) \right\|. \end{aligned} \quad (40)$$

It follows from the above that

$$\left\|\sum_{k=l+1}^{r}\alpha_k\left(T^k u - T^k x\right)\right\| + \left\|\sum_{k=l+1}^{r}\alpha_k\left(T^k x - T^k v\right)\right\| = \left\|\sum_{k=l+1}^{r}\alpha_k\left(T^k u - T^k v\right)\right\|, \tag{41}$$

and since

$$\sum_{k=l+1}^{r}\alpha_k T^k u \neq \sum_{k=l+1}^{r}\alpha_k T^k x \neq \sum_{k=l+1}^{r}\alpha_k T^k v. \tag{42}$$

Then from the strict convexity of $\mathcal{X}$, there exists $\lambda > 0$ such that the following holds:

$$\sum_{k=l+1}^{r}\alpha_k\left(T^k x - T^k v\right) = \lambda\left[\sum_{k=l+1}^{r}\alpha_k\left(T^k u - T^k x\right)\right]. \tag{43}$$

Set $\lambda := \beta/(1-\beta)$ (thus $\beta \in (0,1)$), and equation (43) becomes equivalent to the following:

$$\sum_{k=l+1}^{r}\alpha_k T^k x = \beta\sum_{k=l+1}^{r}\alpha_k T^k u + (1-\beta)\sum_{k=l+1}^{r}\alpha_k T^k v. \tag{44}$$

Consequently, we have

$$\left\|\sum_{k=l+1}^{r}\alpha_k\left(T^k x - T^k v\right)\right\| = \beta\left\|\sum_{k=l+1}^{r}\alpha_k\left(T^k u - T^k v\right)\right\| \tag{45}$$

and

$$\left\|\sum_{k=l+1}^{r}\alpha_k\left(T^k u - T^k x\right)\right\| = (1-\beta)\left\|\sum_{k=l+1}^{r}\alpha_k\left(T^k u - T^k v\right)\right\|. \tag{46}$$

Now, since

$$\begin{aligned}\beta\left\|\sum_{k=l+1}^{r}\alpha_k\left(T^k u - T^k v\right)\right\| &= \left\|\sum_{k=l+1}^{r}\alpha_k\left(T^k x - T^k v\right)\right\|,\\ &\leq \left\|\sum_{k=0}^{l}\alpha_k\left(T^k x - T^k v\right)\right\|,\\ &= c\left\|\sum_{k=l+1}^{r}\alpha_k\left(T^k u - T^k v\right)\right\|,\end{aligned} \tag{47}$$

then, we have that $\beta \leq c$. Similarly, since

$$\begin{aligned}(1-\beta)\left\|\sum_{k=l+1}^{r}\alpha_k\left(T^k u - T^k v\right)\right\| &= \left\|\sum_{k=l+1}^{r}\alpha_k\left(T^k u - T^k x\right)\right\|,\\ &\leq \left\|\sum_{k=0}^{l}\alpha_k\left(T^k u - T^k x\right)\right\|,\\ &= (1-c)\left\|\sum_{k=l+1}^{r}\alpha_k\left(T^k u - T^k v\right)\right\|,\end{aligned} \tag{48}$$

and that gives us $\beta \geq c$ and therefore we must have $\beta = c$. Hence, we have shown that

$$\sum_{k=l+1}^{r}\alpha_k T^k x = c\sum_{k=l+1}^{r}\alpha_k T^k u + (1-c)\sum_{k=l+1}^{r}\alpha_k T^k v, := \sum_{k=l+1}^{r}\alpha_k T^k x - h(T)x, \tag{49}$$

and so we have $h(T) = 0$ or equivalently $x \in A(T)$ and that completes the proof. □

Proof of Theorem 18.

Let $c_k \in [0,1]$ for all $0 \leq k \leq r-1$ and $T^k x := c_k u + (1-c_k)v$, then we have $c_k(u - T^k x) = (1-c_k)(T^k x - v)$, and that also follows that

$$\left\|u - T^k x\right\| = (1-c_k)\|u - v\| \text{ and } \left\|T^k x - v\right\| = c_k\|u - v\|. \tag{50}$$

Now, if $c_0 = 1$, then $x = u$, and this means that $x \in \operatorname{Fix}(T)$ and since by definition $\operatorname{Fix}(T) \subset A(T)$, then it follows that $x \in A(T)$. Similarly, for $c_0 = 0$, then $x = v$ and also follows that $x \in A(T)$. Hence, we may assume that $c_0 \in (0,1)$. First and foremost, we may observe that $u \neq Tx \neq v$, and to see this, we note that if $u = Tx$, then $u = T^{k+1}x$ for all $0 \leq k \leq r-1$ and since by assumption $c_0 \neq 1$, then we have

$$\begin{aligned}D_1(x,v) &= \sum_{k=0}^{r-1} b_k\left\|T^k x - T^k v\right\|, = \sum_{k=0}^{r-1} b_k\left\|T^k x - v\right\|,\\ &= \sum_{k=0}^{r-1} b_k c_k\|u - v\|, < \sum_{k=0}^{r-1} b_k\|u - v\|, \text{since } c_0 \neq 1 (b_0 \neq 0),\\ &= \sum_{k=0}^{r-1} b_k\left\|T^{k+1}x - T^{k+1}v\right\|, = D_1(Tx, Tv) \leq D_1(x,v),\end{aligned} \tag{51}$$

leading to the contradiction that $D_1(x,v) < D_1(x,v)$. In the same vein, if $v = Tx$, then $v = T^{k+1}x$ for all $0 \leq k \leq r-1$. Since by assumption, $c_0 \neq 0$, then we have the following:

$$
\begin{aligned}
D_1(x,v) &= \sum_{k=0}^{r-1} b_k \left\| T^k u - T^k x \right\|, = \sum_{k=0}^{r-1} b_k \left\| u - T^k x \right\|, \\
&= \sum_{k=0}^{r-1} b_k (1-c_k) \|u-v\|, < \sum_{k=0}^{r-1} b_k \|u-v\|, \textit{since}\ c_0 \neq 0 (b_0 \neq 0), \\
&= \sum_{k=0}^{r-1} b_k \left\| T^{k+1} u - T^{k+1} x \right\|, = D_1(Tu, Tx) \leq D_1(u,x),
\end{aligned} \tag{52}
$$

leading to the contradiction that $D_1(u,x) < D_1(u,x)$. Given that T is an $(r,1)$-general higher-order nonexpansive mapping, we have

$$
\begin{aligned}
D_1(u,v) &\leq D_1(u,Tx) + D_1(Tx,v), = D_1(Tu,Tx) + D_1(Tx,Tv), \\
&\leq D_1(u,x) + D_1(x,v), = \sum_{k=0}^{r-1} b_k \left\| T^k u - T^k x \right\| \\
&\quad + \sum_{k=0}^{r-1} b_k \left\| T^k x - T^k v \right\|, = \sum_{k=0}^{r-1} b_k \left\| u - T^k x \right\| \\
&\quad + \sum_{k=0}^{r-1} b_k \left\| T^k x - v \right\|, = \sum_{k=0}^{r-1} b_k (1-c_k) \|u-v\| \\
&\quad + \sum_{k=0}^{r-1} b_k c_k \|u-v\|, = \sum_{k=0}^{r-1} b_k \|u-v\|, \\
&= \sum_{k=0}^{r-1} b_k \left\| T^k u - T^k v \right\|, = D_1(u,v),
\end{aligned} \tag{53}
$$

which implies that $D_1(u,Tx) + D_1(Tx,v) = D_1(u,v)$ or equivalently

$$
\sum_{k=0}^{r-1} b_k \left(\left\| u - T^{k+1} x \right\| + \left\| T^{k+1} x - v \right\| - \|u-v\| \right) = 0. \tag{54}
$$

Now, given that $\|u - T^{k+1}x\| + \|T^{k+1}x - v\| - \|u-v\| \geq 0$ and $b_k > 0(\text{since}\ b_{r-1} = \alpha_r \neq 0)$, then for all $0 \leq k \leq r-1$, we have

$$
\left\| u - T^{k+1} x \right\| + \left\| T^{k+1} x - v \right\| = \|u-v\|. \tag{55}
$$

Since $u \neq Tx \neq v$, then whenever $u \neq T^{k+1}x \neq v$, it follows from the strict convexity of $\mathscr{X}$ that there exists $\lambda_k > 0$ such that

$$
\left(T^{k+1} x - v \right) = \lambda_k \left(u - T^{k+1} x \right). \tag{56}
$$

Now, set $\lambda\_k \coloneqq \beta_k/(1-\beta_k)$ (thus, $\beta_k \in (0,1)$). Then, equation (56) becomes

$$
T^{k+1} x = \beta_k u + (1-\beta_k) v. \tag{57}
$$

Now, when $u = T^{k+1}x$ (respectively $v = T^{k+1}x$) then we choose $\beta_k = 1$ (respectively $\beta_k = 0$). Comparing equation (57) to the definition $T^{k+1}x \coloneqq c_{k+1}u + (1-c_{k+1})v$, it follows that $\beta_k = c_{k+1}$ for all $0 \leq k \leq r-2$. Now, we show that

$$
\beta_{r-1} = \frac{1}{\alpha_r} \sum_{k=0}^{r-1} c_k \alpha_k \ \mathrm{sgn}\ (k). \tag{58}
$$

We observe that $\|T^{k+1}x - v\| = \beta_k \|u-v\|$ and $\|u - T^{k+1}x\| = (1-\beta_k)\|u-v\|$ and hence, we have the following evaluation:

$$
\begin{aligned}
\sum_{k=0}^{r-1} b_k \beta_k \|u-v\| &= \sum_{k=0}^{r-1} b_k \left\| T^{k+1} x - v \right\|, = D_1(Tx,v), = D_1(Tx,Tv), \\
&\leq D_1(x,v) = \sum_{k=0}^{r-1} b_k \left\| T^k x - v \right\|, = \sum_{k=0}^{r-1} b_k c_k \|u-v\|.
\end{aligned} \tag{59}
$$

Hence, we have

$$
\sum_{k=0}^{r-1} b_k (\beta_k - c_k) \leq 0, \tag{60}
$$

since $u \neq v$. Similarly, we have the following evaluation:

$$
\begin{aligned}
\sum_{k=0}^{r-1} b_k (1-\beta_k) \|u-v\| &= \sum_{k=0}^{r-1} b_k \left\| u - T^{k+1} x \right\|, = D_1(u,Tx), \\
&= D_1(Tu,Tx), \leq D_1(u,x), \\
&= \sum_{k=0}^{r-1} b_k \left\| u - T^k x \right\|, = \sum_{k=0}^{r-1} b_k (1-c_k) \|u-v\|.
\end{aligned} \tag{61}
$$

Hence, we have

$$
\sum_{k=0}^{r-1} b_k (\beta_k - c_k) \geq 0, \tag{62}
$$

again because $u \neq v$.

Now, combining equations (60) and (62) and invoking Proposition 14 gives the following:

$$
\begin{aligned}
0 &= \sum_{k=0}^{r-1} b_k (\beta_k - c_k) = \sum_{k=0}^{r-2} b_k (\beta_k - c_k) + b_{r-1} (\beta_{r-1} - c_{r-1}), \\
&= \sum_{k=0}^{r-2} b_k (\beta_k - c_k) + \alpha_r (\beta_{r-1} - c_{r-1}), = \sum_{k=0}^{r-2} b_k (c_{k+1} - c_k) \\
&\quad + \alpha_r (\beta_{r-1} - c_{r-1}), = -b_0 c_0 - \sum_{k=0}^{r-2} c_k (b_k - b_{k-1}) + b_{r-2} c_{r-1} \\
&\quad + \alpha_r (\beta_{r-1} - c_{r-1}), = -\alpha_0 c_0 - \sum_{k=0}^{r-2} c_k \alpha_k \ \mathrm{sgn}\ (k) \\
&\quad + (\alpha_r - \alpha_{r-1} \ \mathrm{sgn}\ (k)) c_{r-1} + \alpha_r (\beta_{r-1} - c_{r-1}), \\
&= \sum_{k=0}^{r-1} c_k \alpha_k \ \mathrm{sgn}\ (k) + \alpha_r c_{r-1} - \alpha_r c_{r-1} + \alpha_r \beta_{r-1}, \\
&= \sum_{k=0}^{r-1} c_k \alpha_k \ \mathrm{sgn}\ (k) + \alpha_r \beta_{r-1}
\end{aligned} \tag{63}
$$

and that gives us

$$\alpha_r\beta_{r-1} = \sum_{k=0}^{r-1} c_k\alpha_k \operatorname{sgn}(k). \tag{64}$$

Since $\alpha_0 \cdot \alpha_r \neq 0$, then, we have

$$\beta_{r-1} = \frac{1}{\alpha_r}\sum_{k=0}^{r-1} c_k\alpha_k \operatorname{sgn}(k), \tag{65}$$

as claimed. Since T is an $(r,1)$-general higher-order nonexpansive mapping, we have that

$$\sum_{k=l+1}^{r} \alpha_k = \sum_{k=0}^{l} \alpha_k, \alpha_r = \sum_{k=0}^{l} \alpha_k - \sum_{k=l+1}^{r-1} \alpha_k, = \sum_{k=0}^{r-1} \alpha_k \operatorname{sgn}(k), \tag{66}$$

and finally, we have

$$1 = \frac{1}{\alpha_r}\sum_{k=0}^{r-1} \alpha_k \operatorname{sgn}(k), \tag{67}$$

since $\alpha_0 \cdot \alpha_r \neq 0$.

Finally, recall that $T^r x = \beta_{r-1}u + (1-\beta_{r-1})v$ and $T^k x = c_k u + (1-c_k)v$ for all $0 \le k \le r-1$, thus we have

$$\begin{aligned} T^r x &= \beta_{r-1}u + (1-\beta_{r-1})v, = \frac{1}{\alpha_r}\sum_{k=0}^{r-1} c_k\alpha_k \operatorname{sgn}(k)u \\ &+ \left(\frac{1}{\alpha_r}\sum_{k=0}^{r-1} \alpha_k \operatorname{sgn}(k) - \frac{1}{\alpha_r}\sum_{k=0}^{r-1} c_k\alpha_k \operatorname{sgn}(k)\right), \\ &= \frac{1}{\alpha_r}\sum_{k=0}^{r-1} \alpha_k \operatorname{sgn}(k)(c_k u + (1-c_k)v), \frac{1}{\alpha_r}\sum_{k=0}^{r-1} \alpha_k \operatorname{sgn}(k)T^k x. \end{aligned} \tag{68}$$

Hence, we have

$$\alpha_r T^r x = \sum_{k=0}^{r-1} \alpha_k T^k x \operatorname{sgn}(k) \tag{69}$$

Observe that

$$\begin{aligned} h(T)x &= \sum_{k=l+1}^{r} \alpha_k T^k x - \sum_{k=0}^{l} \alpha_k T^k x, = \alpha_r T^r x + \sum_{k=l+1}^{r-1} \alpha_k T^k x \\ &- \sum_{k=0}^{l} \alpha_k T^k x, = \alpha_r T^r x - \sum_{k=0}^{r-1} \alpha_k T^k x \operatorname{sgn}(k). \end{aligned} \tag{70}$$

Hence, we have

$$h(T)x = \alpha_r T^r x - \sum_{k=0}^{r-1} \alpha_k T^k x \operatorname{sgn}(k). \tag{71}$$

By combining equations (69) and (71), we get that $h(T)x = 0$ or $x \in A(T)$, and that completes the proof. □

Proof of Theorem 19.

Clearly, if $A_\infty(T) = \emptyset$, then $\operatorname{Fix}(T) = \emptyset$ since by definition we have that $\operatorname{Fix}(T) \subseteq A_\infty(T) \subseteq A(T)$. We first show that $A_\infty(T) = A(T)$. Since by definition, $A_\infty(T) \subseteq A(T)$, we then show that $A(T) \subseteq A_\infty(T)$. Indeed, assume that $T^i x \in A(T)$ for some $i \ge 0$, and then, we have

$$\begin{aligned} h(T)T^i x = 0, \sum_{k=l+1}^{r} \alpha_k T^{i+k}x - \sum_{k=0}^{l} \alpha_k T^{i+k}x = 0, \sum_{k=l+1}^{r} \alpha_k T^{i+k}x \\ = \sum_{k=0}^{l} \alpha_k T^{i+k}x, \alpha_r T^{r+i}x = \sum_{k=0}^{l} \alpha_k T^{i+k}x \\ - \sum_{k=l+1}^{r-1} \alpha_k T^{i+k}x, \alpha_r T^{r+i}x \\ = \sum_{k=0}^{r-1} \alpha_k T^{i+k}x \operatorname{sgn}(k). \end{aligned} \tag{72}$$

Since $\alpha_r \neq 0$, then we have

$$T^{r+i}x = \sum_{k=0}^{r-1} \frac{\alpha_k}{\alpha_r} \operatorname{sgn}(k)T^{i+k}x. \tag{73}$$

By operating both sides of equation (73) under T, we obtain the following:

$$\begin{aligned} T^{r+i+1}x = T\left(T^{r+i}x\right) = T\left(\sum_{k=0}^{r-1} \frac{\alpha_k}{\alpha_r} \operatorname{sgn}(k)T^{i+k}x\right), \\ = \sum_{k=0}^{r-1} \frac{\alpha_k}{\alpha_r} \operatorname{sgn}(k)T^{k+i+1}x, \end{aligned} \tag{74}$$

where the last identity follows because T is *affine* and that $\sum_{k=0}^{r-1}\alpha_k/\alpha_r \operatorname{sgn}(k) = 1$. Hence, we have that $T^{i+1}x \in A(T)$ and so by induction if $x \in A(T)$, we have $T^n x \in A(T)$ for all $n \ge 0$. Hence, $x \in A_\infty(T)$, and that completes the first part of the proof that $A_\infty(T) = A(T)$. Finally, let $q(T) = \sum_{k=0}^{r-1} b_k T^k$; thus the above theorem states that if $h(T)x = 0$, then

$$\frac{1}{\sum_{k=0}^{r-1} b_k} q(T)x \in Fix(T). \tag{75}$$

To see this, observe that (from Proposition 11, noting that here $\lambda = 1$)

$$
\begin{aligned}
q(T)Tx - q(T)x &= \sum_{k=0}^{r-1} b_k\left(T^{k+1}x - T^k x\right), \\
&= b_{r-1}T^r x - b_0 x - \sum_{k=1}^{r-1}(b_k - b_{k-1})T^k x, \\
&= \alpha_r T^r x - \alpha_0 x - \sum_{k=1}^{r-1}(b_k - b_{k-1})T^k x, \\
&= \alpha_r T^r x - \alpha_0 x - \sum_{k=1}^{r-1}\alpha_k T^k x sgn(k), \\
&= \alpha_r T^r x - \alpha_0 x + \sum_{k=l+1}^{r-1}\alpha_k T^k x - \sum_{k=1}^{l}\alpha_k T^k x, \\
&= \alpha_r T^r x + \sum_{k=l+1}^{r-1}\alpha_k T^k x - \alpha_0 x - \sum_{k=1}^{l}\alpha_k T^k x, \\
&= \sum_{k=l+1}^{r}\alpha_k T^k x - \sum_{k=0}^{l}\alpha_k T^k x, = h(T)x.
\end{aligned} \tag{76}
$$

Since T is affine, then we have

$$
T\left(\frac{1}{\sum_{k=0}^{r-1} b_k} q(T)x\right) = \frac{1}{\sum_{k=0}^{r-1} b_k} q(T)Tx, = \frac{1}{\sum_{k=0}^{r-1} b_k}(q(T)x + h(T)x). \tag{77}
$$

and so we have

$$
T\left(\frac{1}{\sum_{k=0}^{r-1} b_k} q(T)x\right) = \frac{1}{\sum_{k=0}^{r-1} b_k} q(T)x. \tag{78}
$$

Once $h(T)x = 0$, and that completes the proof. □

Proof of Theorem 20. For $n \geq 1$, define $T_n : \mathscr{C} \longrightarrow \mathscr{C}$ by

$$
T_n x = T(\gamma_n u + (1-\gamma_n)x), \tag{79}
$$

where $u \in \ker(C)$ is arbitrary and that $\{\gamma_n\}_{n\geq 1} \in (0,1)$ is a null sequence. We show that

$$
T_n^k = \gamma_n \sum_{i=1}^{k}(1-\gamma_n)^{i-1}T^i u + (1-\gamma_n)^k T^k x, \tag{80}
$$

for all $k \geq 1$.

We prove that equation (80) is true by induction. Now, for the case where $k = 1$, by the affiness of T, we have

$$
T_n x = T(\gamma_n u + (1-\gamma_n)x), = \gamma_n Tu + (1-\gamma_n)Tx. \tag{81}
$$

Now, let us assume that equation (80) is true for $k \geq 1$. Note that

$$
\gamma_n \sum_{i=1}^{k}(1-\gamma_n)^{i-1} + (1-\gamma_n)^k = \gamma_n\left(\frac{1-(1-\gamma_n)^k}{\gamma_n} + (1-\gamma_n)^k\right) = 1 \tag{82}
$$

and so by the affiness of T, we have the following evaluation:

$$
\begin{aligned}
T_n^{k+1}x &= T\left(\gamma_n u + (1-\gamma_n)T_n^k x\right), \\
&= \gamma_n Tu + (1-\gamma_n)T\left(\gamma_n \sum_{i=1}^{k}(1-\gamma_n)^{i-1}T^i u + (1-\gamma_n)^k T^k x\right), \\
&= \gamma_n Tu + (1-\gamma_n)\left(\gamma_n \sum_{i=1}^{k}(1-\gamma_n)^{i-1}T^{i+1}u + (1-\gamma_n)^k T^{k+1}x\right), \\
&= \gamma_n \sum_{i=1}^{k+1}(1-\gamma_n)^{i-1}T^i u + (1-\gamma_n)^{k+1}T^{k+1}x,
\end{aligned} \tag{83}
$$

and that completes the proof of equation (80). We have the following evaluation:

$$
\begin{aligned}
T_n^k y - T_n^k x &= \gamma_n \sum_{i=1}^{k}(1-\gamma_n)^{i-1}T^i u + (1-\gamma_n)^k T^k y \\
&\quad - \gamma_n \sum_{i=1}^{k}(1-\gamma_n)^{i-1}T^i u - (1-\gamma_n)^k T^k x, \\
&= (1-\gamma_n)^k\left(T^k y - T^k x\right),
\end{aligned} \tag{84}
$$

and as a result, we have the following:

$$
(1-\gamma_n)^{-k}\left(T_n^k y - T_n^k x\right) = T^k y - T^k x. \tag{85}
$$

Now, by taking norms of equation (85) raised to the power $p (p \geq 1)$, we have

$$
(1-\gamma_n)^{-kp}\left\|T_n^k y - T_n^k x\right\|^p = \left\|T^k y - T^k x\right\|^p. \tag{86}
$$

By Definition 6, we have

$$
\begin{aligned}
&\sum_{k=l+1}^{r}\alpha_k(1-\gamma_n)^{-kp}\left\|T_n^k y - T_n^k x\right\|^p \\
&= \sum_{k=l+1}^{r}\alpha_k\left\|T^k y - T^k x\right\|^p, \leq \sum_{k=0}^{l}\alpha_k\left\|T^k y - T^k x\right\|^p, \\
&= \sum_{k=0}^{l}\alpha_k(1-\gamma_n)^{-kp}\left\|T_n^k y - T_n^k x\right\|^p.
\end{aligned} \tag{87}
$$

Hence, we have

$$\sum_{k=l+1}^{r} \alpha_k (1-\gamma_n)^{-kp} \left\| T_n^k y - T_n^k x \right\|^p \le \sum_{k=0}^{l} \alpha_k (1-\gamma_n)^{-kp} \left\| T_n^k y - T_n^k x \right\|^p. \tag{88}$$

Since T is an (r, p) – general higher-order nonexpansive mapping, then by Proposition 11, we have

$$\begin{aligned} \sum_{k+l}^{r} \alpha_k (1-\gamma_n)^{-kp} &\ge \left(\sum_{k=l+1}^{r} \alpha_k \right) (1-\gamma_n)^{-p(l+1)}, \\ &= \left(\sum_{k=0}^{l} \alpha_k \right) (1-\gamma_n)^{-p(l+1)}, \\ &> \left(\sum_{k=0}^{l} \alpha_k \right) (1-\gamma_n)^{-kp}. \end{aligned} \tag{89}$$

Hence, inequality (88) is an (r, p) – general higher-order contraction mapping, and thus by Theorem 16, T_n has a unique fixed point in $\mathscr{C}$, thus $T_n x_n = x_n$. Now, consider

$$\begin{aligned} \|x_n - Tx_n\| &= \|T_n x_n - Tx_n\|, = \|T(\gamma_n u + (1-\gamma_n)x_n) - Tx_n\|, \\ &= \|\gamma_n Tu + (1-\gamma_n)Tx_n) - Tx_n\|, = \gamma_n \|Tu - Tx_n\|. \end{aligned} \tag{90}$$

Hence, $\|x_n - Tx_n\| \longrightarrow 0$ as $n \longrightarrow \infty$ since $\mathscr{C}$ is bounded and that completes the proof. □

4. Conclusion

As for examples of this map, the immediate examples are algebraic operators (see, for instance, [7, 8]). An algebraic operator is a linear operator satisfying a polynomial identity with scalar coefficients. That is, for any Banach space $\mathscr{X}$ and a given polynomial p, then $T : \mathscr{X} \longrightarrow \mathscr{X}$ is a map such that

$$p(T) = 0. \tag{91}$$

For instance, given the polynomial $p(z) = az^r + a_{r-1}z^{r-1} + \cdots + a_1 z + a_0$, then by the above definition, one obtains the following

$$\begin{aligned} aT^r + a_{r-1}T^{r-1} + \cdots + a_1 T + a_0 I = 0, \\ a(T^r x - T^r y) + a_{r-1}\left(T^{r-1}x - T^{r-1}y\right) + \cdots \\ + a_1(Tx - Ty) + a_0(x - y) = 0. \end{aligned} \tag{92}$$

By taking norm of the above, we have

$$\left\| a(T^r x - T^r y) + a_{r-1}\left(T^{r-1}x - T^{r-1}y\right) + \cdots + a_1(Tx - Ty) + a_0(x - y) \right\| = 0. \tag{93}$$

Which by the subadditivity of norm, we have

$$\begin{aligned} |a| \|T^r x - T^r y\| \le |a_{r-1}| \left\| T^{r-1}x - T^{r-1}y \right\| + \cdots \\ + |a_1| \|Tx - Ty\| + |a_0| \|x - y\|. \end{aligned} \tag{94}$$

which is a higher-order Lipschitz mapping, and hence, a general higher-order Lipschitz mapping.

Algebraic operators are intrinsically interesting and do have good and many applications to other fields in most areas of pure mathematics such as the Connes-Moscovici index theorem for foliated manifolds, algebraic quantum field theory, Novokov conjecture, ordinary and partial differential equations, and Jone's work connecting Von Neumann algebras and geometric topology, which gave rise to a new knot invariant.

Other generalizations (for instance, operators satisfying a polynomial identity with nonscalar coefficients) and their applications can also be found in [9].

Therefore, general higher-order Lipschitz mappings and our current results indeed have the potential of being applied in some mathematical and nonmathematical fields, just like those results mentioned above.

Data Availability

No data was used for this research.

Conflicts of Interest

The author declares that he has no conflicts of interest.

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