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Analytical Solution of Some Higher Degree Equation Via Radicals

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The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

Since the advent of diophantine equations, many different analytic solutions have been formulated for equations with degree $n \leq 4$. However, little has been documented on solvability of equation of degree $n \geq 5$ via the radicals. The majority of recent research seem to have put more focus towards numerical solutions, possibly due to the fact that quintic equations have been proved to be insoluble via the radicals. Let α, β, u and v be any non negative integers. This study delves into realm of analytical solutions of some higher degree equation of the form $\alpha^{6n} + \beta^{6n} = uv$ where u and v are relatively prime.The method of the study involves use of radical solution using case to case basis. In particular,the solution of the equation $\alpha^{6n} + \beta^{6n} = uv$ for $n = 1$ and $n = 2$ where u and v are relatively prime is completely determined.

Keywords: Radicals; Analytical Solution.

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1 Introduction

Since the advent of number theory, many different analytic solutions have been done for equations with degree $n \leq 4$, these solutions are all expressed and connected to the coefficients of the equations. In early 17th and 18th centuries, many scholars devoted their attention in determining the radical solutions to 5^{th} degree equations with very minimal success. Finally, it was shown by Galois that the polynomial equations with degree $n \geq 5$ in their complete form, cannot be solved via the radicals. The above conclusion was partly demonstrated by the researcher P. Ruffini in 1799 [1] and was eventually proved by the Norwegian mathematician N. Abel in 1824 [1], this results received the name Abel-Ruffini theorem. In 1832 E. Galois with his theory gave adequate conditions which establish the solutions of polynomial equations with degree $n \geq 5$ can be found radically [2]. The solution of a given polynomial via the radicals depended on the symmetry group of its roots corresponding to the Cartesian level and the Galois group. Soon after presenting his results Galois passed on after documenting his theory which remained unclaimed. The documents were discovered by J. Liouville in 1843 [2]. In view of the above aforementioned results by Galois that it is impossible to determine the roots of polynomial greater than 5, the interest of mathematicians in this field decreased considerably and many researchers from 19th century onwards concentrated on investigating reliable approximate models for estimating roots for equation whose degree is greater than 5. For studies on polynomial degree greater than 5 , the reader may see [3,4,5,6,7,8,9] and for detailed recap on polynomial less than 5, the reference may be made to [10,11,12,13,14,15,16,17,18]. While mathematicians have made significant advancements in solving polynomial equations through radical expressions, equations of degree five or higher continue to pose a formidable challenge. The age-old quest to develop a general method for solving these equations via radicals remains an ongoing and intriguing pursuit in the realm of mathematics. Researchers are actively exploring new techniques, algorithms, and theoretical frameworks in the hope of one day discovering a comprehensive solution. The unyielding commitment to unraveling the mysteries of higher-degree equations symbolizes the unquenchable thirst for mathematical understanding and the relentless pursuit of knowledge within the mathematical community. This current research delves into the realm of analytical solutions, particularly for higher degree equations greater or equal to multiple of 6. Specifically, the equation of the form $\alpha^{6n} + \beta^{6n} = uv$ where u and v relatively prime is completely characterized via radicals for non-zero distinct integer solution.

2 Main Results

"In the following sections, we present our findings in the form of theorem and proceed to solve particular cases for the exponent n. It's important to note that, in this this research, the condition $\beta > \alpha$ is maintained.

Theorem 2.1. *For any integers α and β, the equation $\alpha^{6n} + \beta^{6n} = uv$ has radical solution if u and v are relatively prime and $n \in \mathbb{N}$. Furthermore, the equation $3\beta^{4n} - 3u\beta^{2n} + u^2 - v = 0$ serves as a representation of the solution of $\alpha^{6n} + \beta^{6n} = uv$.*

Proof. Consider the equation

$$\alpha^{6n} + \beta^{6n} = (\alpha^{2n})^3 + (\beta^{2n})^3 = (\alpha^{2n} + \beta^{2n})(\alpha^{4n} - \alpha^{2n}\beta^{2n} + \beta^{4n})$$

$$= (\alpha^{2n} + \beta^{2n})(\alpha^{4n} - 2\alpha^{2n}\beta^{2n} + \beta^{4n} + \alpha^{2n}\beta^{2n}) = (\alpha^{2n} + \beta^{2n})((\alpha^{2n} - \beta^{2n})^2 + (\alpha^n\beta^n)^2).$$

Suppose that $\alpha^{6n} + \beta^{6n} = uv$. Then, $(\alpha^{2n} + \beta^{2n})((\alpha^{2n} - \beta^{2n})^2 + (\alpha^n\beta^n)^2) = uv \cdots (2.1)$.
Without loss of generality assume $(\alpha^{2n} + \beta^{2n}) = u \cdots (2.2)$ and $((\alpha^{2n} - \beta^{2n})^2 + (\alpha^n\beta^n)^2) = v \cdots (2.3)$. From equation (2.2) , $\alpha^{2n} = u - \beta^{2n}$. Thus, equation (2.3) becomes $(u - \beta^{2n} - \beta^{2n})^2 + (u - \beta^{2n})\beta^{2n} = v \cdots (2.4)$. Expanding equation (2.4) and simplifying further, we obtain, $3\beta^{4n} - 3u\beta^{2n} + u^{2n} = v \cdots (2.5)$ which is a representation of the solution of equation $\alpha^{6n} + \beta^{6n} = uv$.

Notice that, equation (2.5) is a quadratic equation in terms of β^{2n}. Applying radical solution, on case to case basis for different values of n, one obtains the following solutions:

Case $(i), n = 1$ implies $\alpha^6 + \beta^6 = uv$. Thus equation (2.5) reduces to $3\beta^4 - 3u\beta^2 + u^2 = v \cdots (2.6)$. Clearly, equation (2.6) is a quadratic equation in β^2. Applying quadratic formula to solve equation (2.6) for β, the following solution is obtained:

$$\beta_1 = \sqrt{\frac{3u + \sqrt{-3u^2 + 12v}}{6}}, \beta_2 = -\sqrt{\frac{3u + \sqrt{-3u^2 + 12v}}{6}}$$

$$\beta_1 = \sqrt{\frac{3u - \sqrt{-3u^2 + 12v}}{6}}, \beta_2 = -\sqrt{\frac{3u - \sqrt{-3u^2 + 12v}}{6}}$$

To obtain the value α, consider the equation (2.2), $\alpha^2 = u - \beta^2$. so $\alpha = \sqrt{u - \beta^2}$. Substituting for the the values of β, we have

$$\alpha_1 = \sqrt{u - \frac{3u + \sqrt{-3u^2 + 12v}}{6}}, \alpha_2 = -\sqrt{u - \frac{3u + \sqrt{-3u^2 + 12v}}{6}}$$

,

$$\alpha_3 = \sqrt{u - \frac{3u - \sqrt{-3u^2 + 12v}}{6}}, \alpha_4 = -\sqrt{u - \frac{3u - \sqrt{-3u^2 + 12v}}{6}}$$

Hence, for any integers α and β the equation $\alpha^6 + \beta^6 = uv$ where u and v are relatively prime has radical solution.

Case $(ii), n = 2$ implies $\alpha^{12} + \beta^{12} = uv$. Thus equation (2.5) reduces to $3\beta^8 - 3a\beta^4 + u^2 = v \cdots (2.6)$. Clearly, equation (2.6) is a quadratic equation in β^4. Applying quadratic formula to solve equation (2.6) for β, the following solution is obtained:

$$\beta_1 = \sqrt{\sqrt{\frac{3u + \sqrt{-3u^2 + 12v}}{6}}}, \beta_2 = -\sqrt{\sqrt{\frac{3u + \sqrt{-3u^2 + 12v}}{6}}}$$

$$\beta_3 = i^4\sqrt{\sqrt{\frac{3u + \sqrt{-3u^2 + 12v}}{6}}}, \beta_4 = -i^4\sqrt{\sqrt{\frac{3u + \sqrt{-3u^2 + 12v}}{6}}}$$

$$\beta_5 = \sqrt{\sqrt{\frac{3u - \sqrt{-3u^2 + 12v}}{6}}}, \beta_6 = -\sqrt{\sqrt{\frac{3u - \sqrt{-3u^2 + 12v}}{6}}}$$

$$\beta_7 = i^4\sqrt{\sqrt{\frac{3u - \sqrt{-3u^2 + 12v}}{6}}}, \beta_8 = -i^4\sqrt{\sqrt{\frac{3u - \sqrt{-3u^2 + 12v}}{6}}}$$

To obtain the value α, consider the equation (2.2), $\alpha^4 = u - \beta^4$. so $\alpha = \sqrt{u - \beta^4}$. Substituting for the the values of β, we have

$$\alpha_1 = \sqrt{\sqrt{u - \frac{3u + \sqrt{-3u^2 + 12v}}{6}}}, \alpha_2 = -\sqrt{\sqrt{u - \frac{3u + \sqrt{-3u^2 + 12v}}{6}}}$$

,

$$\alpha_3 = i^4\sqrt{\sqrt{u + \frac{3u + \sqrt{-3u^2 + 12v}}{6}}}, \alpha_4 = -i^4\sqrt{\sqrt{u - \frac{3u + \sqrt{-3u^2 + 12v}}{6}}}$$

$$\alpha_5 = \sqrt{\sqrt{u - \frac{3u - \sqrt{-3u^2 + 12v}}{6}}}, \alpha_5 = -\sqrt{\sqrt{u - \frac{3u - \sqrt{-3u^2 + 12v}}{6}}}$$

,

$$\alpha_7 = i^4\sqrt{\sqrt{u + \frac{3u - \sqrt{-3u^2 + 12v}}{6}}}, \alpha_8 = -i^4\sqrt{\sqrt{u - \frac{3u + \sqrt{-3u^2 + 12v}}{6}}}$$

Hence, for any integers α and β the equation $\alpha^{12} + \beta^{12} = uv$ where u and v are relatively prime has radical solution. □

2.1 Applications

Example 2.2. *Let* $\alpha^6 + \beta^6 = 65 = 5 * 13$. *Here,* $u = 5, v = 13, (5, 13) = 1$. *The equation* $3\beta^4 - 3u\beta^2 + u^2 - v = 0$ *is representation of the solution* $\alpha^6 + \beta^6 = 5 * 13$. *Applying the quadratic formula we've*

$$\beta_1 = \sqrt{\frac{3u + \sqrt{-3u^2 + 12v}}{6}} = \sqrt{\frac{3 * 5 + \sqrt{-3 * 5^2 + 12 * 13}}{6}} = 2$$

$$\beta_2 = -\sqrt{\frac{3u + \sqrt{-3u^2 + 12v}}{6}} = -\sqrt{\frac{3 * 5 + \sqrt{-3 * 5^2 + 12 * 13}}{6}} = -2$$

$$\beta_3 = \sqrt{\frac{3u + \sqrt{-3u^2 + 12v}}{6}} = \sqrt{\frac{3 * 5 - \sqrt{-3 * 5^2 + 12 * 13}}{6}} = 1$$

$$\beta_4 = -\sqrt{\frac{3u + \sqrt{-3u^2 + 12v}}{6}} = -\sqrt{\frac{3 * 5 - \sqrt{-3 * 5^2 + 12 * 13}}{6}} = -1$$

$$\alpha_1 = \sqrt{u - \frac{3u + \sqrt{-3u^2 + 12v}}{6}} = \sqrt{5 - \frac{3 * 5 + \sqrt{-3 * 5^2 + 12 * 13}}{6}} = 1,$$

$$\alpha_2 = -\sqrt{u - \frac{3u + \sqrt{-3u^2 + 12v}}{6}} = \sqrt{5 - \frac{3 * 5 + \sqrt{-3 * 5^2 + 12 * 13}}{6}} = -1$$

$$\alpha_3 = \sqrt{u - \frac{3u - \sqrt{-3u^2 + 12v}}{6}} = \sqrt{5 - \frac{3 * 5 - \sqrt{-3 * 5^2 + 12 * 13}}{6}} = 2,$$

$$\alpha_4 = -\sqrt{u - \frac{3u - \sqrt{-3u^2 + 12v}}{6}} = \sqrt{5 - \frac{3 * 5 - \sqrt{-3 * 5^2 + 12 * 13}}{6}} = -2$$

Example 2.3. *Let* $\alpha^6 + \beta^6 = 62281 = 61 * 1021$. *Here,* $u = 61, v = 1021, (61, 1021) = 1$. *The equation* $3\beta^4 - 3u\beta^2 + u^2 - v = 0$ *is representation of the solution* $\alpha^6 + \beta^6 = 61 * 1021$. *Applying the quadratic formula we've*

$$\beta_1 = \sqrt{\frac{3u + \sqrt{-3u^2 + 12v}}{6}} = \sqrt{\frac{3 * 61 + \sqrt{-3 * 61^2 + 12 * 1021}}{6}} = 6$$

$$\beta_2 = -\sqrt{\frac{3u + \sqrt{-3u^2 + 12v}}{6}} = -\sqrt{\frac{3 * 61 + \sqrt{-3 * 61^2 + 12 * 1021}}{6}} = -6$$

$$\beta_3 = \sqrt{\frac{3u + \sqrt{-3u^2 + 12v}}{6}} = \sqrt{\frac{3 * 61 - \sqrt{-3 * 61^2 + 12 * 1021}}{6}} = 5$$

$$\beta_4 = -\sqrt{\frac{3u + \sqrt{-3u^2 + 12v}}{6}} = -\sqrt{\frac{3 * 61 - \sqrt{-3 * 61^2 + 12 * 1021}}{6}} = -5$$

$$\alpha_1 = \sqrt{u - \frac{3u + \sqrt{-3u^2 + 12v}}{6}} = \sqrt{61 - \frac{3 * 61 + \sqrt{-3 * 61^2 + 12 * 1021}}{6}} = 5,$$

$$\alpha_2 = -\sqrt{u - \frac{3u + \sqrt{-3u^2 + 12v}}{6}} = \sqrt{61 - \frac{3 * 61 + \sqrt{-3 * 61^2 + 12 * 1021}}{6}} = -5$$

$$\alpha_3 = \sqrt{u - \frac{3u - \sqrt{-3u^2 + 12v}}{6}} = \sqrt{61 - \frac{3 * 61 - \sqrt{-3 * 61^2 + 12 * 1021}}{6}} = 6,$$

$$\alpha_4 = -\sqrt{u - \frac{3u - \sqrt{-3u^2 + 12v}}{6}} = \sqrt{61 - \frac{3 * 61 - \sqrt{-3 * 61^2 + 12 * 1021}}{6}} = -6$$

Example 2.4. *Let* $\alpha^{12} + \beta^{12} = 535537 = 97 * 5521$*. Here,* $u = 97, v = 5521, (97, 5521) = 1$*. The equation* $3\beta^8 - 3u\beta^4 + u^2 - v = 0$ *is a representation of the solution* $\alpha^{12} + \beta^{12} = 535537 = 97 * 5521.$*. Applying our the general formula, we obtain*

$$\beta_1 = \sqrt{\sqrt{\frac{3u + \sqrt{-3u^2 + 12v}}{6}}} = \sqrt{\sqrt{\frac{3 * 97 + \sqrt{-3 * 97^2 + 12 * 5521}}{6}}} = 3,$$

$$\beta_2 = -\sqrt{\sqrt{\frac{3u + \sqrt{-3u^2 + 12v}}{6}}} = -\sqrt{\sqrt{\frac{3 * 97 + \sqrt{-3 * 97^2 + 12 * 97}}{6}}} = -3,$$

$$\beta_3 = i^4\sqrt{\sqrt{\frac{3u + \sqrt{-3u^2 + 12b}}{6}}} = i^4\sqrt{\sqrt{\frac{3 * 97 + \sqrt{-3 * 97^2 + 12 * 5521}}{6}}} = 3i^4,$$

$$\beta_4 = -i^4\sqrt{\sqrt{\frac{3u + \sqrt{-3u^2 + 12v}}{6}}} = -i^4\sqrt{\sqrt{\frac{3 * 97 + \sqrt{-3 * 97^2 + 12 * 5521}}{6}}} = -3i^4,$$

$$\beta_5 = \sqrt{\sqrt{\frac{3u - \sqrt{-3u^2 + 12v}}{6}}} = \sqrt{\sqrt{\frac{3 * 97 - \sqrt{-3 * 97^2 - 12 * 5521}}{6}}} = 2,$$

$$\beta_6 = -\sqrt{\sqrt{\frac{3u - \sqrt{-3u^2 + 12v}}{6}}} = -\sqrt{\sqrt{\frac{3 * 97 - \sqrt{-3 * 97^2 + 12 * 97}}{6}}} = -2,$$

$$\beta_7 = i^4\sqrt{\sqrt{\frac{3u - \sqrt{-3u^2 + 12v}}{6}}} = i^4\sqrt{\sqrt{\frac{3 * 97 - \sqrt{-3 * 97^2 + 12 * 5521}}{6}}} = 2i^4,$$

$$\beta_8 = -i^4\sqrt{\sqrt{\frac{3u - \sqrt{-3u^2+12v}}{6}}} = -i^4\sqrt{\sqrt{\frac{3*97 - \sqrt{-3*97^2+12*5521}}{6}}} = -2i^4,$$

To obtain the value α, consider the equation (2.2), $\alpha^4 = u - \beta^4$. so $\alpha = \sqrt{u-\beta^4}$. Substituting for the the values of β, we have

$$\alpha_1 = \sqrt{\sqrt{u - \frac{3u + \sqrt{-3u^2+12v}}{6}}} = \sqrt{\sqrt{97 - \frac{3*97 + \sqrt{-3*97^2+12*5521}}{6}}} = 2,$$

$$,\alpha_2 = -\sqrt{\sqrt{u - \frac{3u + \sqrt{-3u^2+12v}}{6}}} = -\sqrt{\sqrt{97 - \frac{3*97 + \sqrt{-3*97^2+12*5521}}{6}}} = -2$$

,

$$\alpha_3 = i^4\sqrt{\sqrt{u - \frac{3u + \sqrt{-3u^2+12v}}{6}}} = i^4\sqrt{\sqrt{97 - \frac{3*97 + \sqrt{-3*97^2+12*5521}}{6}}} = 2i^4,$$

$$\alpha_4 = -i^4\sqrt{\sqrt{u - \frac{3u + \sqrt{-3u^2+12v}}{6}}} = -i^4\sqrt{\sqrt{97 - \frac{3*97 + \sqrt{-3*97^2+12*5521}}{6}}} = -2i^4,$$

$$\alpha_5 = \sqrt{\sqrt{u - \frac{3u - \sqrt{-3u^2+12v}}{6}}} = \sqrt{\sqrt{97 - \frac{3*97 - \sqrt{-3*97^2+12*5521}}{6}}} = 3,$$

$$,\alpha_6 = -\sqrt{\sqrt{u - \frac{3u - \sqrt{-3u^2+12v}}{6}}} = -\sqrt{\sqrt{97 - \frac{3*97 - \sqrt{-3*97^2+12*5521}}{6}}} = -3$$

,

$$\alpha_7 = i^4\sqrt{\sqrt{u + \frac{3u - \sqrt{-3u^2+12v}}{6}}} = i^4\sqrt{\sqrt{97 + \frac{3*97 - \sqrt{-3*97^2+12*5521}}{6}}} = 3i^4,$$

$$\alpha_8 = -i^4\sqrt{\sqrt{u - \frac{3u - \sqrt{-3u^2+12v}}{6}}} = -i^4\sqrt{\sqrt{97 - \frac{3*97 - \sqrt{-3*97^2+12*5521}}{6}}} = -3i^4.$$

Example 2.5. *Let $\alpha^{12} + \beta^{12} = 62281 = 113 * 3361$. Here, $u = 113, v = 3361, (113, 3361) = 1$. The equation $3\beta^8 - 3u\beta^4 + u^2 - v = 0$ is a representation of the solution $\alpha^{12} + \beta^{12} = 62281 = 113 * 3361$.. Applying our the general formula, we obtain*

$$\beta_1 = \sqrt{\sqrt{\frac{3u + \sqrt{-3u^2+12v}}{6}}} = \sqrt{\sqrt{\frac{3*113 + \sqrt{-3*113^2+12*3361}}{6}}} = 8,$$

$$\beta_2 = -\sqrt{\sqrt{\frac{3u + \sqrt{-3u^2+12v}}{6}}} = -\sqrt{\sqrt{\frac{3*113 + \sqrt{-3*113^2+12*3361}}{6}}} = -8,$$

$$\beta_3 = i^4\sqrt{\sqrt{\frac{3u + \sqrt{-3u^2+12b}}{6}}} = i^4\sqrt{\sqrt{\frac{3*113 + \sqrt{-3*113^2+12*3361}}{6}}} = 8i^4,$$

$$\beta_4 = -i^4\sqrt{\sqrt{\frac{3u+\sqrt{-3u^2+12v}}{6}}} = -i^4\sqrt{\sqrt{\frac{3*113+\sqrt{-3*113^2+12*3361}}{6}}} = -8i^4,$$

$$\beta_5 = \sqrt{\sqrt{\frac{3u-\sqrt{-3u^2+12v}}{6}}} = \sqrt{\sqrt{\frac{3*113-\sqrt{-3*113^2-12*3361}}{6}}} = 7,$$

$$\beta_6 = -\sqrt{\sqrt{\frac{3u-\sqrt{-3u^2+12v}}{6}}} = -\sqrt{\sqrt{\frac{3*113-\sqrt{-3*113^2+12*3361}}{6}}} = -7,$$

$$\beta_7 = i^4\sqrt{\sqrt{\frac{3u-\sqrt{-3u^2+12v}}{6}}} = i^4\sqrt{\sqrt{\frac{3*113-\sqrt{-3*113^2+12*3361}}{6}}} = 7i^4,$$

$$\beta_8 = -i^4\sqrt{\sqrt{\frac{3u-\sqrt{-3u^2+12v}}{6}}} = -i^4\sqrt{\sqrt{\frac{3*113-\sqrt{-3*113^2+12*3361}}{6}}} = -7i^4,$$

To obtain the value α*, consider the equation* (2.2)*,* $\alpha^4 = u-\beta^4$*. so* $\alpha = \sqrt{u-\beta^4}$*. Substituting for the the values of* β*, we have*

$$\alpha_1 = \sqrt{\sqrt{u-\frac{3u+\sqrt{-3u^2+12v}}{6}}} = \sqrt{\sqrt{113-\frac{3*113+\sqrt{-3*113^2+12*3361}}{6}}} = 7,$$

$$,\alpha_2 = -\sqrt{\sqrt{u-\frac{3u+\sqrt{-3u^2+12v}}{6}}} = -\sqrt{\sqrt{113-\frac{3*113+\sqrt{-3*113^2+12*3361}}{6}}} = -7$$

,

$$\alpha_3 = i^4\sqrt{\sqrt{u-\frac{3u+\sqrt{-3u^2+12v}}{6}}} = i^4\sqrt{\sqrt{113-\frac{3*113+\sqrt{-3*113^2+12*3361}}{6}}} = 7i^4,$$

$$\alpha_4 = -i^4\sqrt{\sqrt{u-\frac{3u+\sqrt{-3u^2+12v}}{6}}} = -i^4\sqrt{\sqrt{113-\frac{3*113+\sqrt{-3*113^2+12*3361}}{6}}} = -7i^4,$$

$$\alpha_5 = \sqrt{\sqrt{u-\frac{3u-\sqrt{-3u^2+12v}}{6}}} = \sqrt{\sqrt{113-\frac{3*113-\sqrt{-3*113^2+12*3361}}{6}}} = 8,$$

$$,\alpha_6 = -\sqrt{\sqrt{u-\frac{3u-\sqrt{-3u^2+12v}}{6}}} = -\sqrt{\sqrt{113-\frac{3*113-\sqrt{-3*113^2+12*3361}}{6}}} = -8$$

,

$$\alpha_7 = i^4\sqrt{\sqrt{u+\frac{3u-\sqrt{-3u^2+12v}}{6}}} = i^4\sqrt{\sqrt{113+\frac{3*113-\sqrt{-3*113^2+12*3361}}{6}}} = 8i^4,$$

$$\alpha_8 = -i^4\sqrt{\sqrt{u-\frac{3u-\sqrt{-3u^2+12v}}{6}}} = -i^4\sqrt{\sqrt{113-\frac{3*113-\sqrt{-3*113^2+12*3361}}{6}}} = -8i^4$$

3 Conclusion

In conclusion, this research has provided analytical solution for the Diophantine equation $\alpha^{6n} + \beta^{6n} = uv$ where u and v are co-prime, for $n = 1$ and $n = 2$ with α, β being integers. Future research may consider the same diophantine equation for $n \geq 3$ under the representation of the solution $3\beta^{4n} - 3u\beta^{2n} + u^2 - v = 0$. While this research has made good progress in solving the given equation, there is still room for further investigation. Future studies may also delve into extending these analytical method to other classes of Diophantine equations with higher degrees. Moreover, exploring the implication of these methods in applications or number theory would contribute to the broader understanding of diophantine equations.

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Competing Interests

Author has declared no competing interest.

References

[1] Abel NH. Mémoire sur les équations algébriques, ou l'on démontre l'impossibilité de la résolution de l'équation générale du cinquième degré. In S. Ludwig, L. Sophus (Ed.). Euvres Complètes de Niels Henrik Abel, I, 2nd ed. 1881;28-33.

[2] Cavallo A. Galois groups of symmetric sextic trinomials; 2019. arXiv:1902.00965v1 [math.GR]. https://arxiv.org/abs/1902.00965

[3] Giorgos P. Kouropoulos. A combined methodology for approximate estimation of the roots of the general sextic polynomial equation. Research Square; 2021. DOI: https//doi.org/10.21203/rs.3.rs-882192/v2

[4] Mochimaru Y. Solution of Sextic Equations. International Journal of Pure and Applied Mathematics. 2005;23(4):575-583. Available: https://ijpam.eu/contents/2005-23-4/9/9

[5] Najman F. On the Diophantine equation $x^4 \pm y^4 = iz^2$ in gaussian integers. Amer. Math. Monthly. 2010;117(7):637-641.

[6] Najman F.Torsion of elliptic curves over quadratic cyclotomic fields. Math. J. Okayama Univ. 2011;53:75-82.

[7] Par Y. Waring-Golbach problem. Two squares and Higher Powers. Journal de Theorie des Nombres. 2016;791-810.

[8] Ruffini P. Teoria Generale delle Equazioni, in cui si dimostra impossibile la soluzione algebraica delle equazioni generali di grado superiore al quarto [General Theory of equations, in which the algebraic solution of general equations of degree higher than four is proven impossible]. Book on Demand Ltd; 1799. ISBN: 978-5519056762

[9] Tignol JP. Galois' theory of algebraic equations. World Scientific, Louvain, Belgium; 2001. DOI: 10.1142/4628

[10] Amir F, Pooya M, Rahim F. A simple method to solve quartic equations. Australian Journal of Basic and Applied Sciences. 2012;6(6):331-336. ISSN 1991-8178.

[11] Amir F, Nastaran S. A classic new method to solve quartic equations. Applied and Computational Mathematics. 2013;2(2):24-27.
DOI: 10.11648/j.acm.20130202.11

[12] Bombieri E, Bourgain J. A problem on sums of two squares. Internatinal Mathematics Research. 2015;(11):3343-3407.

[13] David A. A partition-theoretic proof of Fermat's two squares theorem. Discrete Mathematics. 2016;339:4:1410–1411.
DOI:10.1016/j.disc.2015.12.002.

[14] Lao H. Some formulae for integer sums of two squares. Journal of Advances in Mathematics and Computer Science. 2022;37(4):53-57.
Article no.JAMCS.87824,ISSN: 2456-9968.

[15] Lao H. Some formulae for integer sums of two squares. Journal of Advances in Mathematics and Computer Science. 2022;38(8):47-52. Article no.JAMCS.101314
ISSN: 2456-9968

[16] Lao H, Zachary K, Kinyanjui J. Some generalized formula for sums of cube. Journal of Advances in Mathematics and Computer Science. 2023;37(4):53-57. Article no.JAMCS.87824
ISSN: 2456-9968

[17] Lao H. On the Diophantine equation $ab(cd+1)+L=u^2+v^2$. Asian Research Journal of Mathematics. 2022;18(9):8-13.
DOI: 10.9734/arjom/2022/v18i930402
Article no.ARJOM.88102, ISSN : 2456-477X

[18] Kimtai B, Lao H. On generalized sum of six, seven and nine cube. Mundi. Science Mundi. 2023;3(1):135-142.
ISSN:2788-5844
Available: http://sciencemundi.net
DOI: https://doi.org/10.51867/scimundi.3.1.14

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