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Research Article

Some New Inequalities Using Nonintegral Notion of Variables

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The object of this paper is to present an extension of the classical Hadamard fractional integral. We will establish some new results of generalized fractional inequalities.

1. Introduction

It is important to note that the integral inequalities play a basic role in statistics, mathematics, sciences, and technology (SMST). As in [1–8], it has proven to be of great importance from the past few decades. The formation of fractional calculus has straight impact on the theory utilizing the solution of various spaces in SMST and to prove its efficacy, various statements and applications of fractional derivatives have been constructed. Authors Riemann–Liouville and Grunwald–Letnikov are well known in this filed. Caputo reformulated the classical statement of the Riemann–Liouville fractional derivative for finding solutions of fractional differential equations using initial conditions. The notion of fractional calculus given by Leibniz was studied by Grunwald–Letnikov in a different structure [9–12].

Recently, in [13–17] and [18], the development between probability theory and fractional calculus was given, and the results of the classical approach were extended. Also, analysis and observations on this direction and several purposes have been found in the concrete problems which include applied mathematics and fluid mechanics as in [2, 18–26], and many others.

As in [27–32], for a function $\mathfrak{g}(\nu) \in L^1([\alpha, \beta])$, the Hadamard fractional integral of order $\kappa \geq 0$ is given as follows:

$$ {}_{H}\mathcal{I}^{\kappa}_{\alpha}[\mathfrak{g}(\nu)] = \frac{1}{\Gamma(\kappa)} \int_{\alpha}^{\nu} \ln\left(\frac{\nu}{u}\right)^{\kappa-1} \mathfrak{g}(u) \frac{du}{u}, \quad 0 < \alpha < \nu \leq \beta, \tag{1} $$

which differs from Riemann-Liouville and Caputo's definition in the sense that the kernel of integral (1) contains a logarithmic function of an arbitrary exponent.

We need the following definition while determining some application, and it is called the Beta function.

As in [22], the Beta function, symbolized by $\beta(l, m)$, is given as

$$ \beta(l, m) = \int_0^1 \tau^{l-m} (1-\tau)^{m-1} d\tau = \frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m)}. \tag{2} $$

The basic notion of generalization of special functions using a kind of new parameter fascinated many researchers and mathematicians. More details about fractional integrals can be found in [33–36] and others as cited in the text. Accordingly, our main scenario in this paper is to extend the idea of a new fractional integration with parameter $\kappa \geq 0$ that generalizes Hadamard fractional integrals.

2. Main Results

In this section, we shall be dealing with the new generalized type of results to random variables of a continuous type of fractional integral order $\kappa \geq 0$.

Definition 1. For a function $\mathfrak{g}(\nu) \in L^1([\alpha, \beta])$, the generalized fractional integral of the Hadamard type with order $\kappa \in \mathbb{R}^+$ is given by

$$ {}_H\mathcal{I}^{-\kappa,\lambda}_{\alpha,\nu}[\mathfrak{g}(\nu)] = \frac{1}{\lambda\Gamma(\kappa)} \int_\alpha^\nu \ln\left(\frac{\nu}{u}\right)^{\kappa/\lambda-1} \mathfrak{g}(u) \frac{du}{u}, \quad 0 < \alpha < \nu \leq \beta, \tag{3} $$

where $\Gamma(\kappa) = \int_0^\infty e^{-u} u^{\kappa-1} du$ represents the Gamma function as can be seen in [37, 38] and many more.

Definition 2. For a r.v. $\mathcal{Z}$ having positive p.d.f. $\mathfrak{g} : [\alpha, \beta] \longrightarrow \mathbb{R}^+$ $(\alpha > 0)$, we define the fractional expectation function of order $\kappa \geq 0$ as

$$ E_{\mathcal{Z},\kappa,\lambda}(\nu) = {}_H\mathcal{I}^{-\kappa,\lambda}_{\alpha,\nu}[\nu\mathfrak{g}(\nu)] = \frac{1}{\lambda\Gamma(\kappa)} \int_\alpha^\nu \left(\ln \frac{\nu}{u}\right)^{\kappa/\lambda-1} u\mathfrak{g}(u) \frac{du}{u}, \tag{4} $$

where $\alpha < \nu < \beta$.

Definition 3. For a r.v. $\mathcal{Z}$ having a positive p.d.f. $\mathfrak{g} : [\alpha, \beta] \longrightarrow \mathbb{R}^+$ $(\alpha > 0)$, the fractional expectation function of $\mathcal{Z} - E(\mathcal{Z})$ of order $\kappa \geq 0$ is given as

$$ E_{\mathcal{Z}-E(\mathcal{Z}),\kappa,\lambda}(\nu) = \frac{1}{\lambda\Gamma(\kappa)} \int_\alpha^\nu \left(\ln \frac{\nu}{u}\right)^{\kappa/\lambda-1} (u - E(\mathcal{Z}))\mathfrak{g}(u) \frac{du}{u}, \tag{5} $$

where $\alpha < \nu < \beta$.

Definition 4. For a r.v. $\mathcal{Z}$ having a positive p.d.f. $\mathfrak{g} : [\alpha, \beta] \longrightarrow \mathbb{R}^+$ $(\alpha > 0)$, the fractional expectation function of order $\kappa \geq 0$ is given as

$$ \begin{aligned} E_{\mathcal{Z},\kappa,\lambda}(\nu) &= {}_H\mathcal{I}^{-\kappa,\lambda}_{1,\nu}[\nu\mathfrak{g}(\nu)] \\ &= \frac{1}{\lambda\Gamma(\kappa)} \int_\alpha^\beta \left(\ln \frac{\nu}{u}\right)^{\kappa/\lambda-1} (u - E(\mathcal{Z}))\mathfrak{g}(u) \frac{du}{u}. \end{aligned} \tag{6} $$

Definition 5. If $E(\mathcal{Z})$ symbolizes the expected value of the r.v. $\mathcal{Z}$ having a positive p.d.f. $\mathfrak{g} : [\alpha, \beta] \longrightarrow \mathbb{R}^+$ with $\alpha > 0$, then the fractional variance function having order κ of $\mathcal{Z}$ is given by

$$ \begin{aligned} \sigma^2_{\mathcal{Z},\kappa,\lambda} &= {}_H\mathcal{I}^{-\kappa,\lambda}_{1,\nu}[(\nu - E(\mathcal{Z})]^2\mathfrak{g}(\nu)] \\ &= \frac{1}{\lambda\Gamma(\kappa)} \int_\alpha^\nu \left(\ln \frac{\nu}{u}\right)^{\kappa/\lambda-1} (u - E(\mathcal{Z}))^2\mathfrak{g}(u) \frac{du}{u}, \end{aligned} \tag{7} $$

where $\alpha < \nu < \beta$.

Definition 6. If $E(\mathcal{Z})$ symbolizes the expected value of the r.v. $\mathcal{Z}$ having a positive p.d.f. $\mathfrak{g} : [\alpha, \beta] \longrightarrow \mathbb{R}^+$ with $\alpha > 0$, then the fractional variance function having order κ of $\mathcal{Z}$ is given by

$$ \sigma^2_{\mathcal{Z},\kappa,\lambda} = \frac{1}{\lambda\Gamma(\kappa)} \int_\alpha^\beta \left(\ln \frac{\nu}{u}\right)^{\kappa/\lambda-1} (u - E(\mathcal{Z}))^2\mathfrak{g}(u) \frac{du}{u}. \tag{8} $$

Now by choosing different values of κ and λ, we have the following remarks.

Remark 7. (R1) Choosing $\lambda = 1$ and $\kappa = 1$, the classical expectation of r.v. $\mathcal{Z}$ will be deduced.

(R2) Choosing $\lambda = 1$ and $\kappa = 1$, the classical variance of r.v. $\mathcal{Z}$ will be deduced.

(R3) Choosing $\lambda = 1$, we reach to the definition of [31].

Theorem 8. *Let the continuous r.v. be $\mathcal{Z}$ with p.d.f. $g : [\alpha, \beta] \longrightarrow \mathbb{R}^+$. Then,*

$$ \sigma^2_{\mathcal{Z},\kappa,\lambda} = E_{\mathcal{Z}^2,\kappa,\lambda} - 2E(\mathcal{Z})E_{\mathcal{Z},\kappa,\lambda} + E(\mathcal{Z})^2_H\mathcal{I}^{-\kappa,\lambda}_{1,\nu}[\mathfrak{g}(\beta)]. \tag{9} $$

for all $\kappa \geq 0$.

Proof. By definition, we have

$$ \begin{aligned} \sigma^2_{\mathcal{Z},\kappa,\lambda} &= \frac{1}{\lambda\Gamma(\kappa)} \int_\alpha^\beta \left(\ln \frac{\nu}{u}\right)^{\kappa/\lambda-1} (u - E(\mathcal{Z}))^2\mathfrak{g}(u) \frac{du}{u} \\ &= \frac{1}{\lambda\Gamma(\kappa)} \int_\alpha^\beta \left(\ln \frac{\nu}{u}\right)^{\kappa/\lambda-1} \left(u^2 - 2uE(\mathcal{Z}) + E(\mathcal{Z})^2\right)\mathfrak{g}(u) \frac{du}{u} \\ &= \frac{1}{\lambda\Gamma(\kappa)} \int_\alpha^\beta \left(\ln \frac{\nu}{u}\right)^{\kappa/\lambda-1} u^2\mathfrak{g}(u) \frac{du}{u} \\ &\quad - 2E(\mathcal{Z}) \frac{1}{\lambda\Gamma(\kappa)} \int_\alpha^\beta \left(\ln \frac{\nu}{u}\right)^{\kappa/\lambda-1} u\mathfrak{g}(u) \frac{du}{u} \\ &\quad + E(\mathcal{Z})^2 \frac{1}{\lambda\Gamma(\kappa)} \int_\alpha^\beta \left(\ln \frac{\nu}{u}\right)^{\kappa/\lambda-1} \mathfrak{g}(u) \frac{du}{u} \\ &= E_{\mathcal{Z}^2,\kappa,\lambda} - 2E(\mathcal{Z})E_{\mathcal{Z},\kappa,\lambda} + E(\mathcal{Z})^2_H\mathcal{I}^{-\kappa,\lambda}_{1,\nu}[\mathfrak{g}(\beta)]. \end{aligned} \tag{10} $$

□

Theorem 9. *Define a r.v. $\mathcal{Z}$ having a p.d.f. $g : [\alpha, \beta] \longrightarrow \mathbb{R}^+$. Then, we have the following inequalities:*

(i) ${}_H\mathcal{I}^{-\kappa,\lambda}_{1,\nu}[\mathfrak{g}(\nu)]\sigma^2_{\mathcal{Z},\kappa,\lambda} - [E_{\mathcal{Z}-E(\mathcal{Z}),\kappa}(\nu)]^2 \leq \|\mathfrak{g}\|^2_\infty[2 (\ln(\nu/u))^{\kappa/\lambda}/\Gamma(\kappa+1){}_H\mathcal{I}^{-\kappa,\lambda}_{1,\nu}[\nu^2] - 2({}_H\mathcal{I}^{-\kappa,\lambda}_{1,\nu}[\nu])^2]$,

holds if $g \in L_\infty[\alpha, \beta]$ and for all $\alpha < \nu \leq \beta$, $\kappa \geq 0$, and $\lambda \geq 0$.

(ii) ${}_H\mathcal{I}^{-\kappa,\lambda}_{1,\nu}[\mathfrak{g}(\nu)]\sigma^2_{\mathcal{Z},\kappa,\lambda} - [E_{\mathcal{Z}-E(\mathcal{Z}),\kappa}(\nu)]^2 \leq 1/2(\nu - \alpha)^2[{}_H\mathcal{I}^{-\kappa,\lambda}_{1,\nu}[\mathfrak{g}(\nu)]$,

holds.

Proof. For the proof of the result, we begin by choosing the function $\mathfrak{H}$ for $x, y \in (\alpha, \nu)$, $\alpha < \nu \le \beta$ as follows:

$$\begin{aligned}\mathfrak{H}(\nu, y) &= (\mathfrak{H}_1(x) - \mathfrak{H}_1(y))(\mathfrak{H}_2(x) - \mathfrak{H}_2(y)) \\ &= \mathfrak{H}_1(x)\mathfrak{H}_2(x) - \mathfrak{H}_1(x)\mathfrak{H}_2(y) - \mathfrak{H}_1(y)\mathfrak{H}_2(\nu) \\ &\quad + \mathfrak{H}_1(y)\mathfrak{H}_2(y),\end{aligned} \tag{11}$$

where $\kappa \ge 0$. □

Now on both sides of (11), we multiply by $((\ln (\nu/x))^{\kappa/\lambda-1}/x\lambda\Gamma(\kappa))\mathfrak{p}(x)$, where the $\mathfrak{p}$ is a function $\mathfrak{p} : [\alpha, \beta] \longrightarrow \mathbb{R}^+$, and then integrating the resulting identity from α to ν, we see

$$\begin{aligned}&\frac{1}{\lambda\Gamma(\kappa)}\int_\alpha^\nu \left(\ln \frac{\nu}{x}\right)^{\kappa/\lambda-1} \mathfrak{p}(x)\mathfrak{H}(x, y)\frac{dx}{x} = {}_H\mathcal{J}_{1,\nu}^{-\kappa,\lambda}[\mathfrak{p}\mathfrak{H}_1\mathfrak{H}_2(\nu)] \\ &\quad - \mathfrak{H}(y)\,{}_H\mathcal{J}_{1,\nu}^{-\kappa,\lambda}[\mathfrak{p}\mathfrak{H}_1(\nu)]\text{-}\mathfrak{H}_1(y)\,{}_H\mathcal{J}_{1,\nu}^{-\kappa,\lambda}[\mathfrak{p}\mathfrak{H}_2(\nu)] \\ &\quad + \mathfrak{H}_1(y)\mathfrak{H}_2(y)\,{}_H\mathcal{J}_{1,\nu}^{-\kappa,\lambda}[\mathfrak{p}(\nu)].\end{aligned} \tag{12}$$

Now multiplying (12) by $((\ln (\nu/y))^{\kappa/\lambda-1}/y\lambda\Gamma(\kappa))\mathfrak{p}(y)$ for $y \in (\alpha, \nu)$, and then integrating the resulting identity over (α, ν) with respect to y, we see

$$\begin{aligned}&\frac{1}{\lambda^2\Gamma^2(\kappa)}\int_\alpha^\nu\int_\alpha^\nu \left(\ln \frac{\nu}{x}\right)^{\kappa/\lambda-1}\left(\ln \frac{\nu}{y}\right)^{\kappa/\lambda-1} \mathfrak{p}(y)\mathfrak{p}(x)\mathfrak{H}(x, y)\frac{dx}{x}\frac{dy}{y} \\ &= 2\,{}_H\mathcal{J}_{1,\nu}^{-\kappa,\lambda}[\mathfrak{p}(\nu)]\,{}_H\mathcal{J}_{1,\nu}^{-\kappa,\lambda}[\mathfrak{p}\mathfrak{H}_1\mathfrak{H}_2(\nu)] \\ &\quad - 2\,{}_H\mathcal{J}_{1,\nu}^{-\kappa,\lambda}[\mathfrak{p}\mathfrak{H}_2(\nu)]\,{}_H\mathcal{J}_{1,\nu}^{-\kappa,\lambda}[\mathfrak{p}\mathfrak{H}_1(\nu)].\end{aligned} \tag{13}$$

Putting $\mathfrak{p}(\nu) = \mathfrak{g}(\nu)$ and $\mathfrak{H}_1(\nu) = \mathfrak{H}_2(\nu) = \nu - E(\mathcal{Z})$, $\nu \in (\alpha, \beta)$ in (13), we see

$$\begin{aligned}&\frac{1}{\lambda^2\Gamma^2(\kappa)}\int_\alpha^\nu\int_\alpha^\nu \left(\ln \frac{\nu}{x}\right)^{\kappa/\lambda-1}\left(\ln \frac{\nu}{y}\right)^{\kappa/\lambda-1} \mathfrak{g}(y)\mathfrak{g}(x)(x-y)^2\frac{dx}{x}\frac{dy}{y} \\ &= 2\,{}_H\mathcal{J}_{1,\nu}^{-\kappa,\lambda}[\mathfrak{g}(\nu)]\,{}_H\mathcal{J}_{1,\nu}^{-\kappa,\lambda}\left[\mathfrak{g}(\nu)\left(\nu - E(X)^2\right)\right] \\ &\quad - 2\,{}_H\mathcal{J}_{1,\nu}^{-\kappa,\lambda}[\mathfrak{g}(\nu)(\nu - E(X))]^2.\end{aligned} \tag{14}$$

But we can also write that as

$$\begin{aligned}&\frac{1}{\lambda^2\Gamma^2(\kappa)}\int_\alpha^\nu\int_\alpha^\nu \left(\ln \frac{\nu}{x}\right)^{\kappa/\lambda-1}\left(\ln \frac{\nu}{y}\right)^{\kappa/\lambda-1} \mathfrak{g}(y)\mathfrak{g}(x)(x-y)^2\frac{dx}{x}\frac{dy}{y} \\ &\le \|\mathfrak{g}\|_\infty^2\frac{1}{\lambda^2\Gamma^2(\kappa)}\int_\alpha^\nu\int_\alpha^\nu \left(\ln \frac{\nu}{x}\right)^{\kappa/\lambda-1}\left(\ln \frac{\nu}{y}\right)^{\kappa/\lambda-1}(x-y)^2\frac{dx}{x}\frac{dy}{y} \\ &\le \|\mathfrak{g}\|_\infty^2\left[2\frac{\ln (\nu/\alpha)^{\kappa/\lambda-1}}{\Gamma(\kappa+1)}\,{}_H\mathcal{J}_{1,\nu}^{-\kappa,\lambda}\left[\nu^2\right] - 2\left({}_H\mathcal{J}_{1,\nu}^{-\kappa,\lambda}[\nu]\right)^2\right].\end{aligned} \tag{15}$$

Consequently, part (i) follows from (14) and (15).

To prove (ii), we can write

$$\begin{aligned}&\frac{1}{\lambda^2\Gamma^2(\kappa)}\int_\alpha^\nu\int_\alpha^\nu \left(\ln \frac{\nu}{x}\right)^{\kappa/\lambda-1}\left(\ln \frac{\nu}{y}\right)^{\kappa/\lambda-1} \mathfrak{g}(y)\mathfrak{g}(x)(x-y)^2\frac{dx}{x}\frac{dy}{y} \\ &\le \sup_{x,y\in[\alpha,\nu]}|x-y|^2\left({}_H\mathcal{J}_{1,\nu}^{-\kappa,\lambda}[\mathfrak{g}(\nu)]\right)^2 \\ &= |x-y|^2\left({}_H\mathcal{J}_{1,\nu}^{-\kappa,\lambda}[\mathfrak{g}(\nu)]\right)^2.\end{aligned} \tag{16}$$

Now using (14) and (16), the part (ii) of the result follows. This completes the proof.

Theorem 10. *Let the continuous r.v. be $\mathcal{Z}$ with p.d.f. $\mathfrak{g} : [\alpha, \beta] \longrightarrow \mathbb{R}^+$. Then*

(i) the inequality

$$\begin{aligned}&{}_H\mathcal{J}_{1,\nu}^{-\kappa,\lambda}[\mathfrak{g}(\nu)]\sigma^2_{\mathcal{Z},\omega,\lambda} + {}_H\mathcal{J}_{1,\nu}^{-\omega,\lambda}[\mathfrak{g}(\nu)]\sigma^2_{\mathcal{Z},\kappa,\lambda} - \left(E_{(\mathcal{Z}-E(\mathcal{Z}),\kappa,\lambda}(\nu)\right) \\ &\times\left(E_{(\mathcal{Z}-E(\mathcal{Z}),\kappa,\omega}(\nu)\right) \le \|\mathfrak{g}\|_\infty^2\left[\frac{(\ln (\nu/\alpha))^{\kappa/\lambda}}{\Gamma(\kappa+1)}\,{}_H\mathcal{J}_{1,\nu}^{-\omega,\lambda}\left[\nu^2\right]\right. \\ &\left.+\frac{\ln (\nu/\alpha)^{\omega/\lambda}}{\Gamma(\kappa+1)}\,{}_H\mathcal{J}_{1,\nu}^{-\kappa,\lambda}\left[\nu^2\right] - \left({}_H\mathcal{J}_{1,\nu}^{-\kappa,\lambda}[\nu]\right)\left({}_H\mathcal{J}_{1,\nu}^{-\omega,\lambda}[\nu]\right)\right],\end{aligned} \tag{17}$$

holds for $\alpha < \nu \le \beta$,$\omega \ge 0$, and $f \in L_\infty([\alpha, \beta])$ and for all $\kappa \ge 0$ and

(ii) the inequality

$$\begin{aligned}&{}_H\mathcal{J}_{1,\nu}^{-\kappa,\lambda}[\mathfrak{g}(\nu)]\sigma^2_{\mathcal{Z},\omega,\lambda} + {}_H\mathcal{J}_{1,\nu}^{-\omega,\lambda}[\mathfrak{g}(\nu)]\sigma^2_{\mathcal{Z},\kappa,\lambda} - \left(E_{(\mathcal{Z}-E(\mathcal{Z}),\kappa,\lambda}(\nu)\right) \\ &\times\left(E_{(\mathcal{Z}-E(\mathcal{Z}),\kappa,\omega}(\nu)\right) \le (\nu-\alpha)^2\left({}_H\mathcal{J}_{1,\nu}^{-\kappa,\lambda}[\mathfrak{g}(\nu)]\right) \\ &\cdot\left({}_H\mathcal{J}_{1,\nu}^{-\omega,\lambda}[\mathfrak{g}(\nu)]\right),\end{aligned} \tag{18}$$

holds for $\alpha < \nu \le \beta$ and $\omega \ge 0$.

Proof. To prove (i), we multiply (12) by $((\ln (\nu/y))^{\omega/\lambda-1}/x\lambda\Gamma(\omega))\mathfrak{p}(y)$ for both sides and get

$$\begin{aligned}&\frac{1}{\lambda^2\Gamma(\kappa)\Gamma(\omega)}\int_\alpha^\nu\int_\alpha^\nu \left(\ln \frac{\nu}{x}\right)^{\kappa/\lambda-1}\left(\ln \frac{\nu}{y}\right)^{\kappa/\lambda-1} \mathfrak{p}(y)\mathfrak{p}(x)\mathfrak{H}(x, y)\frac{dx}{x}\frac{dy}{y} \\ &= {}_H\mathcal{J}_{1,\nu}^{-\kappa,\lambda}[\mathfrak{p}(\nu)]\,{}_H\mathcal{J}_{1,\nu}^{-\omega,\lambda}[\mathfrak{p}(\nu)\mathfrak{H}_1(\nu)\mathfrak{H}_2(\nu)] \\ &\quad + {}_H\mathcal{J}_{1,\nu}^{-\omega,\lambda}[\mathfrak{p}(\nu)]\,{}_H\mathcal{J}_{1,\nu}^{-\kappa,\lambda}[\mathfrak{p}(\nu)\mathfrak{H}_1(\nu)\mathfrak{H}_2(\nu)] \\ &\quad - {}_H\mathcal{J}_{1,\nu}^{-\omega,\lambda}[\mathfrak{p}(\nu)\mathfrak{H}_2(\nu)]\,{}_H\mathcal{J}_{1,\nu}^{-\lambda,\lambda}[\mathfrak{p}(\nu)\mathfrak{H}_1(\nu)] \\ &\quad - {}_H\mathcal{J}_{1,\nu}^{-\kappa,\lambda}[\mathfrak{p}(\nu)\mathfrak{H}_2(\nu)]\,{}_H\mathcal{J}_{1,\nu}^{-\omega,\lambda}[\mathfrak{p}(\nu)\mathfrak{H}_1(\nu)].\end{aligned} \tag{19}$$

□

Putting $\mathfrak{p}(\nu)=\mathfrak{g}(\nu)$, $\mathfrak{H}_1(\nu)=\mathfrak{H}_2(\nu)=\nu-E(\mathscr{X})$, and $\nu\in(\alpha,\beta)$ in (19), we see

$$\begin{aligned}&\frac{1}{\lambda^2\Gamma(\kappa)\Gamma(\omega)}\int_\alpha^\nu\int_\alpha^\nu\left(\ln\frac{\nu}{x}\right)^{\kappa/\lambda-1}\left(\ln\frac{\nu}{y}\right)^{\omega/\lambda-1}\mathfrak{g}(y)\mathfrak{g}(x)(x-y)^2\frac{dx}{x}\frac{dy}{y}\\&={}_H\mathscr{I}_{1,\nu}^{-\kappa,\lambda}[\mathfrak{g}(\nu)]_H\mathscr{I}_{1,\nu}^{-\omega,\lambda}\left[\mathfrak{g}(\nu)\left(\nu-E(\mathfrak{X})^2\right)\right]\\&\quad+{}_H\mathscr{I}_{1,\nu}^{-\omega,\lambda}[\mathfrak{g}(\nu)]\times{}_H\mathscr{I}_{1,\nu}^{-\kappa,\lambda}\left[\mathfrak{g}(\nu)\left(\nu-E(\mathfrak{X})^2\right)\right]\\&\quad-2{}_H\mathscr{I}_{1,\nu}^{-\kappa,\lambda}[\mathfrak{g}(\nu)(\nu-E(\mathfrak{X}))]_H\mathscr{I}_{1,\nu}^{-\omega,\lambda}[\mathfrak{g}(\nu)(\nu-E(\mathfrak{X}))].\end{aligned}\tag{20}$$

But we can also write

$$\begin{aligned}&\frac{1}{\lambda^2\Gamma(\kappa)\Gamma(\omega)}\int_\alpha^\nu\int_\alpha^\nu\left(\ln\frac{\nu}{x}\right)^{\kappa/\lambda-1}\left(\ln\frac{\nu}{y}\right)^{\omega/\lambda-1}\mathfrak{g}(y)\mathfrak{g}(x)(x-y)^2\frac{dx}{x}\frac{dy}{y}\\&\le\|\mathfrak{g}\|_\infty^2\frac{1}{\lambda^2\Gamma^2(\kappa)}\int_\alpha^\nu\int_\alpha^\nu\left(\ln\frac{\nu}{x}\right)^{\kappa/\lambda-1}\left(\ln\frac{\nu}{y}\right)^{\kappa/\lambda-1}(x-y)^2\frac{dx}{x}\frac{dy}{y}\\&\le\|\mathfrak{g}\|_\infty^2\left[\frac{(\ln(\nu/\alpha))^{\kappa/\lambda-1}}{\Gamma(\kappa+1)}{}_H\mathscr{I}_{1,\nu}^{-\kappa,\lambda}\left[\nu^2\right]+\frac{(\ln(\nu/\alpha))^{\omega/\lambda}}{\Gamma(\omega+1)}{}_H\mathscr{I}_{1,\nu}^{-\omega,\lambda}\left[\nu^2\right]\right.\\&\quad\left.-2\left({}_H\mathscr{I}_{1,\nu}^{-\kappa,\lambda}[\nu]\right)\left({}_H\mathscr{I}_{1,\nu}^{-\omega,\lambda}[\nu]\right)\right].\end{aligned}\tag{21}$$

Consequently, part (i) of the result follows from (20) and (21).

To prove part (ii), we use the truth that $\sup_{\nu,y\in[\alpha,\nu]}|x-y|^2=(x-\alpha)^2$ and get

$$\begin{aligned}&\frac{1}{\lambda^2\Gamma(\kappa)\Gamma(\omega)}\int_\alpha^\nu\int_\alpha^\nu\left(\ln\frac{\nu}{x}\right)^{\kappa/\lambda-1}\left(\ln\frac{\nu}{y}\right)^{\omega/\lambda-1}\mathfrak{g}(y)\mathfrak{g}(x)(x-y)^2\frac{dx}{x}\frac{dy}{y}\\&\le(x-\alpha)^2\left({}_H\mathscr{I}_{1,\nu}^{-\kappa,\lambda}[\mathfrak{g}(\nu)]\right)\left({}_H\mathscr{I}_{1,\nu}^{-\omega,\lambda}[\mathfrak{g}(\nu)]\right).\end{aligned}\tag{22}$$

Consequently, part (ii) of the result follows by employing (20) and (21).

3. Applications and Examples

Application 11. Consider the positive functions $\mathfrak{g}$ and $\mathfrak{h}$ on $[\alpha,\beta]$ such that for every $u>\alpha$, $0<{}_H\mathscr{I}_{1,\nu}^{-\kappa,\lambda}\mathfrak{g}^m,{}_H\mathscr{I}_{1,\nu}^{-\kappa,\lambda}\mathfrak{h}^m<\infty$ with

$$0<K\le\frac{\mathfrak{g}(u)}{\mathfrak{h}(u)}\le L<\infty,u\in[\alpha,\beta],\tag{23}$$

where $m\ge1$; then, for every $\kappa>0$, we have

$$\begin{aligned}&\left({}_H\mathscr{I}_{1,\nu}^{-\kappa,\lambda}(\mathfrak{g}^m(u)\right)^{2/m}+\left({}_H\mathscr{I}_{1,\nu}^{-\kappa,\lambda}(\mathfrak{h}^m(u)\right)^{2/m}\\&\ge\left[\frac{(K+1)(L+1)}{L}-2\right]\left[{}_H\mathscr{I}_{1,\nu}^{-\kappa,\lambda}(\mathfrak{g}^m(\nu)\right]^{1/m}\left({}_H\mathscr{I}_{1,\nu}^{-\kappa,\lambda}\mathfrak{h}^m(\nu)\right)^{1/m}.\end{aligned}\tag{24}$$

Solution 12. From (23), we see

$$\begin{aligned}\frac{1}{L}&\le\frac{\mathfrak{h}(u)}{\mathfrak{g}(u)}\Rightarrow\left(\frac{1}{L}+1\right)^m\\&\le\left(\frac{\mathfrak{h}(u)}{\mathfrak{g}(u)}+1\right)^m\Rightarrow(L+1)^m\mathfrak{g}^m(u)\\&\le L^m(\mathfrak{h}+\mathfrak{g})^m(u).\end{aligned}\tag{25}$$

In a similar way, we see

$$K\le\frac{\mathfrak{g}(u)}{\mathfrak{h}(u)}\Rightarrow(K+1)^m\mathfrak{g}^m(u)\le(\mathfrak{h}+\mathfrak{g})^m(u).\tag{26}$$

Consequently, multiplying these equations by $(\ln(\nu/u))^{\kappa/\lambda-1}/u\lambda\Gamma(\kappa)$ for $u\in(\alpha,\nu)$ and then integrating the resulting identity over (α,ν) with respect to u yield

$$\left({}_H\mathscr{I}_{1,\nu}^{-\kappa,\lambda}(\mathfrak{g}^m(u))\right)^{1/m}\le\frac{L}{L+1}((\mathfrak{g}+\mathfrak{h})^m(u))^{1/m},\tag{27}$$

$$\left({}_H\mathscr{I}_{1,\nu}^{-\kappa,\lambda}((\mathfrak{h}^m(u))\right)^{\frac{1}{m}}\le\frac{1}{K+1}((\mathfrak{g}+\mathfrak{h})^m(u)))^{\frac{1}{m}}.\tag{28}$$

Now on multiplying (27) and (28), we see

$$\begin{aligned}&\frac{(K+1)(L+1)}{L}\left({}_H\mathscr{I}_{1,\nu}^{-\kappa,\lambda}((\mathfrak{g}^m(u))\right)^{1/m}\left({}_H\mathscr{I}_{1,\nu}^{-\kappa,\lambda}(\mathfrak{h}^m(u))\right)^{1/m}\\&\le\left({}_H\mathscr{I}_{1,\nu}^{-\kappa,\lambda}((\mathfrak{g}+\mathfrak{h})^m(u))\right)^{2/m}.\end{aligned}\tag{29}$$

Consequently, the result follows by using Minkowski's integral inequality on the right-hand side of (29).

Example 1. Consider the function $\mathfrak{g}(u)=(\ln(u/\alpha))^{\eta/\lambda-1}$, we see

$${}_H\mathscr{I}_{\alpha,u}^{-\kappa,\lambda}\left(\ln\frac{u}{\alpha}\right)^{\eta/\lambda-1}=\frac{1}{\lambda\Gamma(\kappa)}\int_\alpha^\nu\left(\ln\frac{u}{w}\right)^{\kappa/\lambda-1}\left(\ln\frac{w}{\alpha}\right)^{\eta/\lambda-1}\frac{dw}{w}.\tag{30}$$

Choosing $\tau=\ln(u/w)/\ln(u/\alpha)$, for $\tau\in(\alpha,\beta]$ with η, $\kappa>0$, we see with the help of (2) that

$$\begin{aligned}{}_H\mathscr{I}_{\alpha,u}^{-\kappa,\lambda}\left(\ln\frac{u}{\alpha}\right)^{\kappa/\lambda-1}&=\left(\ln\frac{u}{\alpha}\right)^{(\kappa+\eta)/\lambda-1}\frac{1}{\lambda\Gamma(\kappa)}\int_0^1(1-\tau)^{\eta/\lambda-1}(1-\tau)^{\kappa/\lambda-1}d\tau\\&=\left(\ln\frac{\nu}{w}\right)^{\kappa/\lambda-1}\frac{\beta(\kappa/\lambda,\eta/\lambda)}{\lambda\Gamma(\kappa)}.\end{aligned}\tag{31}$$

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that there is no conflict of interest.

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