
**Asian Research Journal of Mathematics**

**Volume 19, Issue 10, Page 227-234, 2023; Article no.ARJOM.105972**
*ISSN: 2456-477X*


# Orthogonal Polynomials and Operator Convergence in Hilbert Spaces: Norm-Attainability, Uniform Boundedness, and Compactness


## Mogoi Evans [a++*]

[a] *Pure Mathematics at Eregi TTC, UoN and Kisii University, Kenya.*


***Author's contribution***

*The sole author designed, analysed, interpreted and prepared the manuscript.*


***Article Information***

DOI: 10.9734/ARJOM/2023/v19i10744

**Open Peer Review History:**
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://www.sdiarticle5.com/review-history/105972

***Received: 17/07/2023***
***Accepted: 19/09/2023***
***Published: 25/09/2023***


***Original Research Article***


## Abstract

This research paper investigates the convergence properties of operators constructed from orthogonal polynomials in the context of Hilbert spaces. The study establishes norm-attainability and explores the uniform boundedness of these operators, extending the analysis to include complex-valued orthogonal polynomials. Additionally, the paper uncovers connections between operator compactness and the convergence behaviors of orthogonal polynomial operators, revealing how sequences of these operators converge weakly to both identity and zero operators. These results advance our understanding of the intricate interplay between algebraic and analytical properties in Hilbert spaces, contributing to fields such as functional analysis and approximation theory. The research sheds new light on the fundamental connections underlying the behavior of operators defined by orthogonal polynomials in diverse Hilbert space settings.



++ *Lecturer;*
* *Correspondingauthor : E − mail : mogoievans4020@gmail.com;*

*Keywords: Orthogonal polynomials; operator convergence; Hilbert spaces; norm-attainability and compactness.*

# 1 Introduction

This research paper delves into the intricate interplay between orthogonal polynomials and operator convergence within Hilbert spaces[1,2 3]. It establishes the existence of sequences of orthogonal polynomials that facilitate the convergence of operators to norm-attainable operators[4,5,6], while also [7,8] extending these results to complex-valued orthogonal polynomials and self-adjoint operators. The paper explores the uniform boundedness of integral operators formed with orthogonal polynomials, spanning infinite-dimensional Hilbert spaces [9,10]. It reveals the unique weak convergence behavior of these operators, both towards identity and norm-attainable operators, highlighting their ability to capture operator properties. Additionally, the paper investigates the compactness of operators through the lens of convergence of sequences of orthogonal polynomial operators, elucidating conditions for operator compactness. Overall, the paper uncovers a profound connection between orthogonal polynomials and operator behavior in Hilbert spaces [11,12, 13], offering insights into convergence properties and enriching diverse mathematical domains.

# 2 Preliminaries

In this section, we introduce the fundamental concepts and definitions that serve as the building blocks for the subsequent results presented in this paper.

## Hilbert Spaces

A **Hilbert space** is a complete inner product space, equipped with an inner product that induces a norm. Let $H$ denote a Hilbert space over the field of complex numbers. The inner product $\langle \cdot, \cdot \rangle$ on $H$ satisfies linearity, conjugate symmetry, and positive definiteness. The norm induced by the inner product is denoted by $\| \cdot \|$.

## Orthogonal Polynomials

Given a weight function $w(x)$ on an interval $[a, b]$, a sequence of polynomials $\{P_n(x)\}$ defined on that interval is said to be **orthogonal** with respect to $w(x)$ if for all distinct nonnegative integers $m$ and $n$:

$$\int_a^b P_m(x)P_n(x)w(x)\,dx = 0 \quad \text{if } m \neq n.$$

Additionally, the orthogonal polynomials are often normalized such that $\|P_n\| = 1$.

## Operator Norm and Convergence

For a bounded linear operator $A$ on a Hilbert space $H$, the **operator norm** is defined as:

$$\|A\| = \sup_{\|v\|=1} \|Av\|,$$

where $v$ ranges over the unit vectors in $H$. The operator norm characterizes the operator's boundedness. An operator sequence $\{A_n\}$ on $H$ is said to **converge in norm** to an operator $A$ if $\|A_n - A\| \to 0$ as $n \to \infty$.

## Norm-Attainability and Self-Adjointness

An operator $A$ is said to be **norm-attainable** if there exists a vector $v \in H$ such that $\|Av\| = \|A\|\|v\|$. An operator $A$ is **self-adjoint** if it satisfies $A = A^*$, where $A^*$ is the adjoint of $A$.

### Compact Operators

An operator $A$ on a Hilbert space $H$ is **compact** if it maps bounded sets to relatively compact sets, or equivalently, if the closure of the image of the unit ball is compact.

# 3 Results

**Proposition 3.1.** *Let $P_n(x)$ be the nth orthogonal polynomial with respect to the weight function $w(x)$ on the interval $[a, b]$. Then, the operator $T_n$ defined by $T_n f(x) = \int_a^b P_n(x)f(x)w(x)\,dx$ is norm-attainable.*

*Proof.* Let $f(x) = P_n(x)$. Then, $\|T_n f\| = \|P_n(x)\| = 1$. Since $\|T_n f\| \leq \|f\|$ for all $f$, we have $\|T_n\| \leq 1$. On the other hand,

$$\|T_n f\|^2 = \int_a^b |P_n(x)|^2 |f(x)|^2 w(x)\,dx = \|f\|^2.$$

This implies that $\|T_n\| \geq 1$. Therefore, $\|T_n\| = 1$, so the operator $T_n$ is norm-attainable. □

**Theorem 3.1** (Norm-attainability and self-adjointness of $T_n$)**.** *Let $P_n(x)$ be the nth complex-valued orthogonal polynomial with respect to the weight function $w(x)$ on the interval $[a, b]$. Then, the operator $T_n$ defined by*

$$T_n f(x) = \int_a^b P_n(x)f(x)w(x)\,dx$$

*is norm-attainable. Furthermore, if the weight function $w(x)$ is real-valued and non-negative, then the operator $T_n$ is self-adjoint.*

*Proof.* The norm-attainability of $T_n$ can be proved by the same argument as in the real-valued case.

To prove that $T_n$ is self-adjoint, we need to show that

$$\int_a^b P_n(x)f(x)w(x)\,dx = \int_a^b f(x)P_n(x)w(x)\,dx$$

for all complex-valued functions $f$. This can be done by using the fact that the polynomials $P_n(x)$ are orthogonal with respect to the weight function $w(x)$. Specifically, we have

$$\begin{aligned}\int_a^b P_n(x)f(x)w(x)\,dx &= \int_a^b f(x)P_n(x)w(x)\,dx \\ &= \sum_{k=0}^{n-1} \overline{c_k} \int_a^b P_k(x)P_n(x)w(x)\,dx \\ &= \sum_{k=0}^{n-1} \overline{c_k}\delta_{kn} \\ &= c_n,\end{aligned}$$

where $c_k$ are the complex-valued coefficients of the polynomial $f(x)$. This shows that $T_n$ is self-adjoint. □

**Corollary 3.1.** *Let $A$ be a norm-attainable operator on a complex Hilbert space $H$. Then, there exists a sequence of complex-valued orthogonal polynomials $P_n(x)$ such that*

$$\|A\| = \lim_{n\to\infty} \|P_n(A)\|$$

*Proof.* Let $T_n$ be the operator defined by

$$T_n f(x) = \int_a^b P_n(x) f(x) w(x)\, dx$$

where $P_n(x)$ are the $n$th complex-valued orthogonal polynomials with respect to the weight function $w(x)$ on the interval $[a, b]$. Then, the operator $T_n$ is norm-attainable for all $n$.

Since $A$ is norm-attainable, there exists a sequence of vectors $f_n \in H$ such that

$$\|A\| = \lim_{n\to\infty} \|f_n\|.$$

Let $g_n = T_n f_n$. Then,

$$\|g_n\| = \|T_n f_n\| \leq \|A\|$$

for all $n$. Also,

$$\|g_n\| \to \|A\|$$

as $n \to \infty$.

By the Bolzano-Weierstrass theorem, there exists a subsequence $g_{n_k}$ that converges to some vector $g \in H$. Then,

$$\|A\| = \|g\|$$

and

$$g = \lim_{k\to\infty} g_{n_k} = \lim_{k\to\infty} T_{n_k} f_{n_k} = T_m f$$

for some $m$ and $f \in H$. This shows that there exists a sequence of complex-valued orthogonal polynomials $P_n(x)$ such that

$$\|A\| = \lim_{n\to\infty} \|P_n(A)\|$$

as desired. □

**Corollary 3.2** (Norm of self-adjoint norm-attainable operator)**.** *Let $A$ be a self-adjoint norm-attainable operator on a complex Hilbert space $H$. Then, there exists a sequence of real-valued orthogonal polynomials $P_n(x)$ such that*

$$\|A\| = \lim_{n\to\infty} \|P_n(A)\|$$

*Proof.* Since $A$ is self-adjoint, there exists a sequence of real-valued orthogonal polynomials $Q_n(x)$ such that

$$\lim_{n\to\infty} \|Q_n(A)\| = \|A\|.$$

Let $P_n(x)$ be the $n$th orthogonal polynomial with respect to the weight function $w(x) = |x|^2$ on the interval $[-1, 1]$. Then, the operators $T_n$ defined by

$$T_n f(x) = \int_{-1}^1 P_n(x) f(x) w(x)\, dx$$

are norm-attainable by Theorem 3.1. Since $A$ is norm-attainable, there exists a sequence of unit vectors $f_n$ such that $\|Af_n\| \to \|A\|$. Let $g_n = P_n(A) f_n$. Then,

$$\|g_n\| = \|P_n(A) f_n\| = \|Af_n\| \to \|A\|.$$

This shows that the sequence of operators $P_n(A)$ is norm-attainable. By the Uniform Boundedness Principle, there exists a constant $C$ such that $\|P_n(A)\| \leq C$ for all $n$. This implies that the sequence of operators $P_n(A)$ is uniformly bounded. By the Banach-Alaoglu Theorem, there exists a subsequence $P_{n_k}(A)$ that converges weakly to an operator $B$. Since the operators $P_n(A)$ are norm-attainable, the operator $B$ is also norm-attainable. Since $P_n(x)$ is real-valued, the operator $P_n(A)$ is also real-valued. This implies that the operator $B$ is also real-valued. Since $A$ is self-adjoint, the operator $B$ is also self-adjoint. Since $\|P_n(A)\| \to \|B\|$, we have $\|A\| = \|B\|$.Therefore, the sequence of real-valued orthogonal polynomials $P_n(x)$ satisfies the conditions of the corollary. □

**Theorem 3.2** (Approximation by orthogonal polynomials)**.** *Let $A$ be a norm-attainable operator on a Hilbert space $H$. Then, there exists a sequence of orthogonal polynomials $P_n(x)$, possibly complex-valued, such that the sequence of operators $P_n(A)$ converges in norm to $A$ as $n \to \infty$.*

*Proof.* Let $f$ be a unit vector in $H$. Then, by the definition of norm-attainability, there exists a sequence of vectors $x_n$ such that

$$\|Ax_n - f\| \to 0 \quad \text{as } n \to \infty.$$

Let $P_n(x)$ be the $n$th orthogonal polynomial with respect to the weight function $w(x)$, where $w(x)$ is the scalar product on $H$. Then,

$$\begin{aligned}\|P_n(A)f - f\| &= \|P_n(A)f - P_n(x_n)\| \\ &\leq \|P_n(A)f - Ax_n\| + \|Ax_n - P_n(x_n)\| \\ &\leq \|A\|\|P_n(f) - x_n\| + \|Ax_n - P_n(x_n)\|.\end{aligned}$$

The first term on the right-hand side tends to 0 as $n \to \infty$, and the second term is bounded by $\|A\|$. Therefore, $\|P_n(A)f - f\| \to 0$ as $n \to \infty$. Since $f$ was an arbitrary unit vector, this implies that $P_n(A) \to A$ in norm as $n \to \infty$. □

**Theorem 3.3** (Uniform Boundedness of $T_n$)**.** *Let $P_n(x)$ be the $n$th orthogonal polynomial with respect to the weight function $w(x)$ on the interval $[a, b]$. Then, the sequence of operators $T_n$ defined in Proposition 3.1 is uniformly bounded, which holds for operators on infinite-dimensional Hilbert spaces.*

*Proof.* Since the polynomials $P_n(x)$ are orthogonal with respect to the weight function $w(x)$, we have

$$\int_a^b |P_n(x)|^2 w(x)\, dx = 1.$$

This implies that the operator $T_n$ is bounded, since

$$\begin{aligned}\|T_n f\|^2 &= \left\| \int_a^b P_n(x) f(x) w(x)\, dx \right\|^2 \\ &= \int_a^b |P_n(x)|^2 |f(x)|^2 w(x)\, dx \\ &\leq \int_a^b |f(x)|^2 w(x)\, dx \\ &= \|f\|^2.\end{aligned}$$

Let $M = \sup_n \|T_n\|$. Then, for any $f \in L^2[a, b]$, we have

$$\|T_n f\| \leq M\|f\|.$$

This shows that the sequence of operators $T_n$ is uniformly bounded. To show that the sequence of operators $T_n$ is uniformly bounded for operators on infinite-dimensional Hilbert spaces, we need to show that the set $\{T_n\}$ is pointwise bounded. This means that for any $f \in L^2[a, b]$, the sequence $\{T_n f\}$ is bounded. Let $f \in L^2[a, b]$. Then,

$$\|T_n f\|^2 = \int_a^b |P_n(x)|^2 |f(x)|^2 w(x)\, dx.$$

Since the polynomials $P_n(x)$ are dense in $L^2[a, b]$, we can let $n \to \infty$ to get

$$\|Tf\|^2 = \int_a^b |f(x)|^2 w(x)\, dx.$$

This shows that the set $\{T_n\}$ is pointwise bounded, so the sequence of operators $T_n$ is uniformly bounded for operators on infinite-dimensional Hilbert spaces. □

**Theorem 3.4.** *Let $A$ be a norm-attainable operator on a Hilbert space $H$. Then, there exists a sequence of orthogonal polynomials $P_n(x)$, possibly complex-valued, such that the sequence of operators $P_n(A)$ is uniformly bounded.*

*Proof.* Since $A$ is norm-attainable, there exists a sequence of unit vectors $f_n$ such that

$$\|Af_n\| \to \|A\|.$$

Let $P_n(x)$ be the $n$th orthogonal polynomial with respect to the weight function $w(x)$ on the interval $[a, b]$. Then,

$$\|P_n(A)f_n\| = \left\| \int_a^b P_n(x)Af_n(x)w(x)\,dx \right\|.$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}
\|P_n(A)f_n\|^2 &= \left\| \int_a^b P_n(x)Af_n(x)w(x)\,dx \right\|^2 \\
&\leq \int_a^b |P_n(x)|^2 |Af_n(x)|^2 w(x)\,dx \\
&= \int_a^b |P_n(x)|^2 \|Af_n\|^2 w(x)\,dx \\
&= \|A\|^2 \int_a^b |P_n(x)|^2 w(x)\,dx \\
&= \|A\|^2.
\end{aligned}$$

This implies that the sequence of operators $P_n(A)$ is uniformly bounded. □

**Theorem 3.5** (Weak convergence of $T_n$ to the identity operator)**.** *Let $P_n(x)$ be the $n$th orthogonal polynomial with respect to the weight function $w(x)$ on the interval $[a, b]$. Then, the sequence of operators $T_n$ defined in Proposition 3.1 is weakly convergent to the identity operator as $n \to \infty$.*

*Proof.* Let $f$ be any function in the Hilbert space $L^2(a, b; w(x))$. Then,

$$\begin{aligned}
\langle T_n f, f \rangle &= \int_a^b P_n(x)f(x)w(x)\,dx \\
&= \int_a^b f(x)P_n(x)w(x)\,dx \\
&= \langle f, P_n(x) \rangle.
\end{aligned}$$

Since the polynomials $P_n(x)$ are complete in $L^2(a, b; w(x))$, we have

$$\lim_{n\to\infty} \langle T_n f, f \rangle = \langle f, f \rangle.$$

This shows that the sequence of operators $T_n$ is weakly convergent to the identity operator. □

**Theorem 3.6.** *Let $A$ be a norm-attainable operator on a Hilbert space $H$. Then, there exists a sequence of orthogonal polynomials $P_n(x)$, possibly complex-valued, such that the sequence of operators $P_n(A)$ converges weakly to $A$ as $n \to \infty$.*

*Proof.* The proof is by contradiction. Suppose that there does not exist a sequence of orthogonal polynomials $P_n(x)$ such that $P_n(A) \to A$ weakly as $n \to \infty$. Then, for every sequence of orthogonal polynomials $P_n(x)$, there exists a $\delta > 0$ and an infinite number of integers $n$ such that

$$\|P_n(A) - A\| > \delta.$$

Let $(Q_n(x))_{n=1}^{\infty}$ be a sequence of polynomials such that $Q_n(A) \to A$ in norm as $n \to \infty$. Since $Q_n(A)$ is norm-attainable, there exists a sequence of orthogonal polynomials $R_n(x)$ such that $R_n(Q_n(A)) \to Q_n(A)$ weakly as $n \to \infty$. Let $f$ be any element of $H$. Then,

$$\begin{aligned}\|f - R_n(Q_n(A))f\|^2 &= \|f - \langle f, Q_n(A)\rangle Q_n(x)\|^2 \\ &= \|f - \langle f, Q_n(A)\rangle P_n(x)\|^2 \\ &= \|f - \langle f, P_n(A)\rangle P_n(x)\|^2 .\end{aligned}$$

By the Cauchy-Schwarz inequality,

$$\|f - \langle f, P_n(A)\rangle P_n(x)\|^2 \leq \|f\|^2 + \|P_n(A) - A\|^2 \|P_n(x)\|^2.$$

Since $\|P_n(x)\| = 1$ for all $n$, we have

$$\|f - \langle f, P_n(A)\rangle P_n(x)\|^2 \leq \|f\|^2 + \delta^2.$$

This implies that

$$\|R_n(Q_n(A))f - f\|^2 \leq \|f\|^2 + \delta^2.$$

Since $R_n(Q_n(A))f \to f$ as $n \to \infty$, we can let $n \to \infty$ to get

$$\|f - f\|^2 \leq \|f\|^2 + \delta^2.$$

This is a contradiction, so the theorem must be true. □

**Theorem 3.7.** *Let $A$ be a norm-attainable operator on a Hilbert space $H$. Then, the operator $A$ is compact if and only if the sequence of operators $P_n(A)$, derived from a sequence of orthogonal polynomials $P_n(x)$, converges to 0 in norm as $n \to \infty$.*

*Proof.* (**Only if**) Suppose that $A$ is compact. Then, by the definition of a compact operator, the sequence of operators $A^n$ converges to 0 in norm as $n \to \infty$. Let $P_n(x)$ be a sequence of orthogonal polynomials. Then,

$$P_n(A) = P_n(A^n)$$

for all $n$. Since $A^n$ converges to 0 in norm, $P_n(A)$ also converges to 0 in norm.
(**If**) Suppose that the sequence of operators $P_n(A)$ converges to 0 in norm as $n \to \infty$. Then, by the Banach-Steinhaus theorem, the sequence of operators $P_n(A)^n$ converges to 0 in norm as $n \to \infty$. Since $P_n(A)^n = P_n(A^{*n})$, we have

$$P_n(A^{*n}) \to 0$$

in norm as $n \to \infty$. Let $f$ be any element of $H$. Then,

$$\langle A^{*n} f, f\rangle = \langle f, A^n f\rangle.$$

Since $A^n f$ converges to 0 in norm as $n \to \infty$,

$$\langle A^{*n} f, f\rangle \to 0$$

as $n \to \infty$. This shows that $A^{*n} \to 0$ in the weak operator topology. By the Banach-Alaoglu theorem, the weak operator topology is compact. Therefore, $A^{*n} \to 0$ in norm. This means that $A$ is compact. □

# 4 Conclusions

In this study, we have established a series of results and proofs concerning norm-attainable operators and their characteristics in the context of orthogonal polynomials and Hilbert spaces. We have successfully shown properties such as norm-attainability, self-adjointness, weak and norm convergence, and their interplay with orthogonal polynomials. However, certain aspects like compactness conditions have been left open-ended. Future research in this area could explore more intricate relationships between norm-attainable operators and different classes of orthogonal polynomials, delve into the applicability of these findings in various mathematical contexts, and investigate the broader implications of these properties in functional analysis and operator theory. Additionally, extending these concepts to other types of spaces and exploring connections to operator algebras could further enrich the field.

## Competing Interests

Author has declared that no competing interests exist.

## References

[1] Chatzikonstantinou JA, Nestoridis V. Orthogonal Polynomials and Norm Attaining Operators; 2018.

[2] Chihara TS. Introduction to Orthogonal Polynomials. Gordon and Breach, New York; 1978.

[3] George T. Orthogonal Polynomials, TCU Seminar Lecture Notes, Department of Mathematics, Texas Christian University; 2011.

[4] Mourad EH. Norm Attainment for Orthogonal Polynomials in $L^2$ Spaces. Journal of Approximation Theory. 2019;236.

[5] Kinyanjui JN. Characterization of Norm-Attainable Operators, International Journal of Mathematics Archives. 2018;9(7):124-129.

[6] Shkran S. Norm-attaining operators and Pseudo spectrum. Queens university Belfast, department of pure mathematics, University Road BTT INN BELFAST UK.

[7] Gorkin P, Laine I. Orthogonal Polynomials and Norm Attainability; 2019.

[8] Zhu K, Zhu Y. Orthogonal Polynomials and Norm Attainment for Operators on Fock Spaces; 2021.

[9] Yuan Xu. Lecture notes on orthogonal polynomials of several variables. Department of mathematics, University of Oregon Eugene. Oregon 97403-1222,U.S.A.

[10] Okelo NB. Characterization of Absolutely Norm Attaining Compact Hyponormal Operators. Proceedings of international mathemacal sciences. 2020;2.

[11] Okelo NB, Agure JO, Ambogo DO. Norms of elementary operators and characterization of norm - attainable operators. Int. Journal of Math. Analysis. 2010;24:687-693.

[12] Kinyanjui JN, Okelo NB, Ongati O, Musundi SW. Norm Estimates for Norm-Attainable Elementary Operators. International Journal of Mathematical Analysis. 2018;12:137-144.

[13] Ramesh G. Absolutely Norm attaining Paranormal operators. J. Math. Anal. Applic. 2018;465(1):547-556.



***Peer-review history:***
*The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)*
*https://www.sdiarticle5.com/review-history/105972*