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Some Fixed Point Theorems in $e-$ Complete E-Metric Space

K. Manoj [a*] and R. Jahir Hussain [a]

[a] *PG & Research Department of Mathematics, Jamal Mohamed College (Autonomous) (Affiliated to Bharathidasan University), Tiruchirappalli-620020, Tamilnadu, India.*

Authors' contributions

This work was carried out in collaboration between both authors. Author KM designed the study, performed the theoretical analysis, wrote the protocol, and wrote the first draft of the manuscript. Author RJH managed the analyses of the study and managed the theoretical reviews. Both authors read and approved the final manuscript.

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Abstract

In this study, we present fixed-point results for various contractions in $\mathtt{E}-$ Metric Space along with several corollaries. Finding a fixed point result in $\mathtt{E}-$ Metric Space can also be done via implicit relations. In the investigation of our primary result stands, all the theorems will be in non-solid cone and we cannot assume interior point is non-empty.

Keywords: $\mathtt{E}-$ *metric space; e− complete; non-solid cones; semi-interior points.*

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**Corresponding author: E-mail: manojguru542@gmail.com;*

1 Introduction and Preliminaries

The notion of Cone metric space introduced by Huang and Zhang [1] replacing the real numbers by an ordered Banach space, also they discovered fixed point theorems in cone metric space using contraction mapping. In 2008, Rezapour and Hamlbarani [2] redefined fixed point result without assumptions of normality of cone in Banach space. Al-Rawashdeh et.al. [3] introduced the concept of $\mathtt{E}-$ metric space in 2012. TVS-valued cone Banach space were defined by Mehmood et al.[4] in 2014.

Typically, the Banach space $\mathtt{E}$ was taken into account with a defined order in relation to the positive solid cone $\mathtt{E}^+$ of $\mathtt{E}$, i.e., by assuming that interior of $\mathtt{E}^+$ is not empty. The non-solid cones were just briefly mentioned in a few results [5, 6]. When non-solid cones are involved, [6] took into account the quasi interior points of P rather than interior points.

In 2017, Polyrakis [7] introduced the notions of semi-interior points and see some other article related to cone and $\mathtt{E}-$ metric space [8, 9, 6]. Mehmood et al. [10] generalized the Banach,Kannan and Chatterjea contractions in the setting of $\mathtt{E}-$ metric space via non-solid cones. some other reference related to fixed point theorems see [11, 12].

In this article, we develope Ciric type and some other different kinds of contraction in the setting of $\mathtt{E}-$ metric space. Also finally define some implicit relation in the setting of $\mathtt{E}-$ metric space.

A normed space $\mathtt{E}$ with $\|\cdot\|$, its ordered by the positive cone $\mathtt{E}^+$, then $u, v \in \mathtt{E}$, *such that* $u \preceq v$ iff $v - u \in \mathtt{E}^+$.

Definition 1.1. [3] An ordered space $\mathtt{E}$ is a vector space over $\mathbb{R}$ with a partial order relation $\preceq$ such that

(OS_1) $\forall u, v \ \& \ z \in \mathtt{E}, if \ u \preceq v \implies u + z \preceq v + z$,

(OS_2) $\forall \upsilon \in \mathbb{R}^+ \ \& \ u \in \mathtt{E}, u \succeq 0_{\mathtt{E}}, \upsilon u \succeq 0_{\mathtt{E}}$.

Likewise, if $\mathtt{E}$ contains a norm, it is referred to as a normed ordered space.

Definition 1.2. A normed ordered space $\mathtt{X}$'s positive cone $\mathtt{E}^+$ is known as:

1. Normal, $\exists \ N > 0 \ni$
$$0 \leq u \leq v \implies \|u\| \leq N\|v\|$$
$\forall u, v \in \mathtt{X}$.
2. Solid, then $\mathtt{E}^+$ has non-empty interior,
3 Reflexive, if and only if $\mathtt{E}^+ \cap U$ is weakly compact, where U is the unit ball in $\mathtt{X}$,
4 Strongly reflexive, if and only if $\mathtt{E}^+ \cap U$ is compact.

In the above definition, reflexive and strongly reflexive cones can be defined in [6].

Definition 1.3. [3] Given that $\mathtt{E}$ is an ordered space over real scalars, let $\mathtt{X}$ be a nonempty set. For all u, v and z in $\mathtt{X}$, an ordered $\mathtt{E}$-metric on $\mathtt{X}$, denoted by the $\mathtt{E}-$ valued function $p^{\mathtt{E}} : \mathtt{X} \times \mathtt{X} \to \mathtt{E}$, we have

(e_1) $p^{\mathtt{E}}(u, v) \geq 0_{\mathtt{E}}$ and $p^{\mathtt{E}}(u, v) = 0_{\mathtt{E}}$ if and only if $u = v$,

(e_2) $p^{\mathtt{E}}(u, v) = p^{\mathtt{E}}(v, u)$,

(e_3) $p^{\mathtt{E}}(u, v) \preceq p^{\mathtt{E}}(u, z) + p^{\mathtt{E}}(z, v)$.

Then the tuple $(\mathtt{X}, p^{\mathtt{E}})$ is identified as $\mathtt{E}$-metric space.

The definition that follows presumes that int $(\mathtt{E}^+)$ is not empty.

Definition 1.4. [10] Consider the $\mathtt{E}-$ metric space of an normed ordered space $\mathtt{E}$ and (u_n) in $\mathtt{X}$ is called convergent to $u \in \mathtt{X}$ *then,* $\forall\ c \in$ int $(\mathtt{E}^+)$, $\exists$ a natural number $\mathbb{N}$,

$$p^{\mathtt{E}}(u_n, u) \ll c$$

$\forall n \geq \mathbb{N}$,

$$\lim_{n\to\infty} u_n = u,$$

or $u_n \to u$. The sequence (u_n) is called Cauchy, $\forall c \in$ int $(\mathtt{E}^+)$, $\exists$ a natural number N_1 such that

$$p^{\mathtt{E}}(u_n, u_m) \ll c$$

$\forall n, m \leq N_1$

Definition 1.5. [7] Let $\mathtt{E}$ be a normed space with orders determined by the positive cone $\mathtt{E}^+$. We will represent the zero of $\mathtt{E}$ by the symbol $0_{\mathtt{E}}$. Let $U = \{u \in \mathtt{E} \ni \|u\| \leq 1\}$ represent the closed unit ball of $\mathtt{E}$, and by U^+ we refer to the positive portion of the unit ball formed by the set

$$U^+ = U \cap \mathtt{E}^+.$$

Definition 1.6. [7] A point $u_0 \in \mathtt{E}^+$ is a semi-interior point of $\mathtt{E}^+$ if $\exists \rho > 0$ such that

$$u_0 - \rho U^+ \subseteq \mathtt{E}^+$$

Clearly any interior point of $\mathtt{E}^+$ is semi-interior point. The Collection of all semi-interior points of $\mathtt{E}^+$ as indicated by $(\mathtt{E}^+)^{\ominus}$,

$$\forall u, v \in \mathtt{E}^+, u \lll v \iff v - u \in (\mathtt{E}^+)^{\ominus}$$

.

Example 1.1. *[7] Let $\mathtt{X_n}$ be the space of $\mathbb{R}^2$ ordered by the pointwise ordering and with norm $\|u_n\|_n$ having unit ball. The polygon D_n of $\mathbb{R}^2$ with vertices $(1,0),(0,1),(-n,n),(-1,0),(0,-1),(n,-n)$. The norm is given by*

$$\|(u,v)\|_n = \begin{cases} |u| + |v|, & \text{if } uv \geq 0 \\ \max\{|u|,|v|\} - \frac{n-1}{n}\min\{|u|,|v|\}, & \text{if } uv < 0. \end{cases}$$

Suppose that $\mathtt{E}$ is the space of all the sequences $(u_n) \ni u_n = (u_1^n, u_2^n) \in \mathtt{u_n}$ with $\|u_n\|_n \leq \gamma(u) \forall n \in \mathbb{N}$ and $\gamma > 0$, depends on u. Assume that $\mathtt{E}$ is ordered by cone $\mathtt{E}^+ = \{u = (u_n) \in \mathtt{X} : u_n \in \mathbb{R}^2_+ \text{ for any } n\}$ and equipped with the norm

$$\|u\|_\infty = \sup_{n\in\mathbb{N}} \|u_n\|_n.$$

Also, suppose that $\mathtt{X} = \mathtt{E}^+ - \mathtt{E}^+$ is the subspace of $\mathtt{E}$ generated by $\mathtt{E}^+$ and ordered by $\mathtt{E}^+$. Let $1 = (v_n)$, where $v_n = (1,1)$ for any $n \in \mathbb{N}$. Then 1 is not an interior point of $\mathtt{E}^+$. For any $m \in \mathbb{N}$ and $v = (v_n) \in \mathtt{X} \ni$,

$$v = (v_n) = \begin{cases} \zeta_n = (-2,2), & \text{for } n = m \\ v_n = (0,0), & \text{for } n \neq m \end{cases}$$

It is easy to show $\|v\|_\infty = \frac{2}{m}$ and $1 + v \notin \mathtt{E}^+$, even though any $u \in \mathtt{E}^+$ is not an interior point of $\mathtt{E}^+$, therefore $\mathtt{E}^+$ has an empty interioc. The positive part of closed unit ball is

$$U^+ = \{u \in \mathtt{E}^+ : \|u\|_\infty \leq 1\}.$$

For any $u = (u_n) \in U^+$ it is easy to see that $\|u_n\|_n = u_n^1 + u_n^2 \leq 1$,

$$1 - U^+ \subseteq \mathtt{E}^+$$

and 1 is a semi-interior point of $\mathtt{E}^+$. Also the space $\mathtt{X}$ is not complete using Proposition 2.4 of [7].

Example 1.2. *(In Example 2.7 of [7] and Example 5.9 of [8]) it has been shown that a strong reflexive cone* $\mathtt{E}^+$ *of* $L_1([0,1])$ *exists which generate a dense subspace* $\mathtt{X}$ *of* $L_1([0,1])$ *, i.e.,*

$$\mathtt{X} = \mathtt{E}^+ - \mathtt{E}^+$$

and

$$L_1([0,1]) = \bar{\mathtt{X}}.$$

Let $V = co\left(\left(B^+(0,1)\right) \cup \left(-B^+(0,1)\right)\right)$ *and* E_1^+ *be a set of positive elements of* $L_1([0,1])$ *generated by the set* $\Omega = 3v + V$ *, where* $v = \sum_{k=1}^{\infty} \alpha^{k-1} e_k$ *for* $\alpha \in (0,1)$ *, where* $\{e_i\}$ *is the set of standard normalized basis. It has been shown in [7], that the cone* E^+ *has empty interior but has semi-interior points.*

Hence every interior point of the cone $\mathtt{E}^+$ *is the semi-interior point if int* $\left(\mathtt{E}^+\right)$ *is nonempty.*

Proposition 1.1. *[7] Any point that is semi-interior to a closed cone* $\mathtt{E}^+$ *in a complete ordered normed space* $\mathtt{E}$ *is also an interior point of* $\mathtt{E}^+$*.*

Definition 1.7. [10] Assume that $\left(\mathtt{E}^+\right)^{\ominus}$ is non-empty and $\mathtt{E}$ is the ordered normed space with $\left(\mathtt{X}, p^{\mathtt{E}}\right)$ be an $\mathtt{E}-$ metric space. For any $(u_n) \in \mathtt{X}$ and $u \in \mathtt{X}$, then

(i) (u_n) e-converges to u when for every $e \ggg 0_E$, $\exists$ a natural number $u_n \xrightarrow{e} u$

(ii) (u_n) is an e-Cauchy sequence when for every $e \ggg 0_{\mathtt{E}}$, $\exists$ $k \in \mathbb{N}$ $\ni$ $p^{\mathtt{E}}(u_n, u_m) \ll e$ for all $n, m \geq k$.

(iii) $\left(\mathtt{X}, p^{\mathtt{E}}\right)$ is e-complete if every e-Cauchy sequence is e-convergent.

Theorem 1.3. *[10] Let* $G : \mathtt{X} \to \mathtt{X}$ *be a mapping and* $\left(\mathtt{X}, p^{\mathtt{E}}\right)$ *be an e-complete* $\mathtt{E}$*-metric space with positive closed cone* $\mathtt{E}^+$ *satisfying*

$$p^{\mathtt{E}}(\mathtt{G}u, \mathtt{G}v) \preceq a p^{\mathtt{E}}(u, v)$$

$\forall u, v \in \mathtt{X}$ *and* $a \in [0,1)$ *. Then* $\mathtt{G}$ *has a unique fixed point in* $\mathtt{X}$ *and for each* $u \in \mathtt{X}$ *, the sequence of iteration* $(\mathtt{G}^{\mathtt{n}} u)_{n>0}$ *converges to a fixed-point of* G *.*

Corollary 1.4. *[10] Consider an e-complete* $\mathtt{E}$*-metric space* $\left(\mathtt{X}, p^{\mathtt{E}}\right)$ *with positive closed cone* $\mathtt{E}^+ \ni \left(\mathtt{E}^+\right)^{\ominus} \neq \varphi$ *. For* $e \ggg 0_{\mathtt{E}}, u_0 \in \mathtt{X}$ *, set* $B(u_0, e) = \left\{u \in \mathtt{X} : p^{\mathtt{E}}(u, v) \preceq e\right\}$ *and* $\mathtt{G} : \mathtt{X} \to \mathtt{X}$ *be a mapping such that*

$$p^{\mathtt{E}}(\mathtt{G}u, \mathtt{G}v) \preceq a p^{\mathtt{E}}(u, v)$$

$\forall u, v \in B(u_0, e)$ *, where* $a \in [0,1)$ *and* $p^{\mathtt{E}}(u_0, \mathtt{G}u_0) \preceq (1-a)e$ *. Then* $\mathtt{G}$ *has a unique fixed point in* $B(u_0, e)$ *.*

Corollary 1.5. *[10] Let* $\left(\mathtt{X}, p^{\mathtt{E}}\right)$ *be an* $e-$ *complete* $\mathtt{E}$*-metric space with positive closed cone* $\mathtt{E}^+$ *such that* $\left(\mathtt{E}^+\right)^{\ominus} \neq \varphi$*. Let for some* $n \in \mathbb{N}$*, the mapping* $\mathtt{G} : \mathtt{X} \to \mathtt{X}$ *satisfies*

$$p^{\mathtt{E}}(\mathtt{G}^{\mathtt{n}} u, \mathtt{G}^{\mathtt{n}} v) \preceq a p^{\mathtt{E}}(u, v)$$

for all $u, v \in \mathtt{X}$ *, where* $a \in [0,1)$ *is a constant. Then* $\mathtt{G}$ *has a unique fixed point in* $\mathtt{X}$ *.*

Theorem 1.6. *[10] Let* $\left(\mathtt{X}, p^{\mathtt{E}}\right)$ *be an* $e-$ *complete* $\mathtt{E}$*-metric space with positive closed cone* $\mathtt{E}^+$ *such that* $\left(\mathtt{E}^+\right)^{\ominus} \neq \varphi$*. A mapping* $\mathtt{G} : \mathtt{X} \to \mathtt{X}$ *satisfies*

$$p^{\mathtt{E}}(\mathtt{G}u, \mathtt{G}v) \preceq \theta\left[p^{\mathtt{E}}(\mathtt{G}u, u) + p^{\mathtt{E}}(\mathtt{G}v, v)\right]$$

$\forall u, v \in \mathtt{X}$ *and some* $\theta \in \left[0, \frac{1}{2}\right)$*. Then* $\mathtt{G}$ *has a unique fixed point in* $\mathtt{X}$*, and for each* $u \in \mathtt{X}$*,* $(\mathtt{G}^{\mathtt{n}} u)_{n \geq 0}$ *converges to a fixed point of* $\mathtt{X}$*.*

2 Main Result

Theorem 2.1. *Consider an be a normed space* $\mathtt{E}$ *and it's ordered by a closed positive cone* $\mathtt{E}^+$. *A function* $\mathtt{G} : \mathtt{X} \to \mathtt{X}$, *where* $\mathtt{X}$ *is an* $e-$ *complete* $\mathtt{E}-$ *metric space with positive closed cone* $\mathtt{E}^+$ *such that* $(\mathtt{E}^+)^{\ominus} \neq \varphi$, *it satisfies*

$$\min\{p^{\mathtt{E}}(\mathtt{G}u, \mathtt{G}v), p^{\mathtt{E}}(u, \mathtt{G}u), p^{\mathtt{E}}(v, \mathtt{G}v)\} - \min\{p^{\mathtt{E}}(u, \mathtt{G}v), p^{\mathtt{E}}(v, \mathtt{G}u)\} \preceq ap^{\mathtt{E}}(u, v) \tag{2.1}$$

$\forall u, v \in \mathtt{X}$ & $a \in (0,1)$ *and* $p^{\mathtt{E}}(u, u_n) \ll (1-a)\frac{\tau}{m}$. *Then* $\{G^n u\}$ *converges to a fixed point of* $\mathtt{G}$

Proof. Consider the iterative sequence

$$u_{n+1} = \mathtt{G}u_n = G^n u_0$$

where $u_0 \in \mathtt{X}$, with $u_n \neq u_{n+1} \forall n \in \mathbb{N}$ for $n \in \mathbb{N}$. Substitute $u = u_n, v = u_{n+1}$ in equation (2.1), we get

$$\min\{p^{\mathtt{E}}(\mathtt{G}u_n, \mathtt{G}u_{n+1}), p^{\mathtt{E}}(u_n, \mathtt{G}u_n), p^{\mathtt{E}}(u_{n+1}, \mathtt{G}u_{n+1})\} - \min\{p^{\mathtt{E}}(u_n, \mathtt{G}u_{n+1}), p^{\mathtt{E}}(u_{n+1}, \mathtt{G}u_n)\} \preceq ap^{\mathtt{E}}(u_n, u_{n+1})$$

$$\min\{p^{\mathtt{E}}(u_{n+1}, \mathtt{G}u_n), p^{\mathtt{E}}(u_n, u_{n+1}), p^{\mathtt{E}}(u_{n+1}, u_n)\} - \min\{p^{\mathtt{E}}(u_n, u_n), p^{\mathtt{E}}(u_{n+1}, u_{n+1})\} \preceq ap^{\mathtt{E}}(u_n, u_{n+1}) \tag{2.2}$$

Consider the LHS of equation (2.2),

$$\min\{p^{\mathtt{E}}(u_{n+1}, \mathtt{G}u_n), p^{\mathtt{E}}(u_n, u_{n+1}), p^{\mathtt{E}}(u_{n+1}, u_n)\} - \min\{p^{\mathtt{E}}(u_n, u_n), p^{\mathtt{E}}(u_{n+1}, u_{n+1})\} = \min\{p^{\mathtt{E}}(u_{n+1}, u_{n+2}), p^{\mathtt{E}}(u_n, u_{n+1})\} - \min\{p^{\mathtt{E}}(u_n, u_{n+2}), p^{\mathtt{E}}(u_{n+1}, u_{n+1})\}$$

Suppose if $\min\{p^{\mathtt{E}}(u_{n+1}, u_{n+2}), p^{\mathtt{E}}(u_n, u_{n+1})\} = p^{\mathtt{E}}(u_n, u_{n+1})$,
From equation (2.2)

$$p^{\mathtt{E}}(u_n, u_{n+1}) \preceq ap^{\mathtt{E}}(u_n, u_{n+1})$$

It is not possible, since $a \in (0,1)$

$$\therefore \min\{p^{\mathtt{E}}(u_{n+1}, u_{n+2}), p^{\mathtt{E}}(u_n, u_{n+1})\} = p^{\mathtt{E}}(u_{n+1}, u_{n+2}).$$

Again for equation (2.2), we get

$$p^{\mathtt{E}}(u_{n+1}, u_n) \preceq ap^{\mathtt{E}}(u_n, u_{n+1}). \tag{2.3}$$

Now,

$$\begin{aligned} p^{\mathtt{E}}(u_n, u_{n+1}) &= p^{\mathtt{E}}(\mathtt{G}u_n, \mathtt{G}u_{n-1}) \\ &\preceq ap^{\mathtt{E}}(u_n, u_{n-1}) \\ &\vdots \\ &\preceq a^n p^{\mathtt{E}}(u_1, u_0). \end{aligned}$$

Now for $n > m$, consider

$$\begin{aligned} p^{\mathtt{E}}(u_m, u_n) &\preceq p^{\mathtt{E}}(u_m, u_{m+1}) + p^{\mathtt{E}}(u_{m+1}, u_{m+2}) + \cdots + p^{\mathtt{E}}(u_{n-1}, u_n) \\ &\preceq \left(a^m + a^{m+1} + \cdots + a^{n-1}\right) p^{\mathtt{E}}(u_1, u_0) \\ &\preceq a^m \left(1 + a + a^2 + \cdots + a^{n-m-1}\right) p^{\mathtt{E}}(u_1, u_0) \\ &\preceq a^m \left(\frac{1 - a^{n-m}}{1-a}\right) p^{\mathtt{E}}(u_1, u_0) \end{aligned}$$

For any $\tau \ggg 0_{\mathtt{E}}, \exists \rho > 0$ such that $\tau - \rho U^+ \subseteq \mathtt{E}^+$ and $n_1 \in \mathbb{N}$ such that $a^m \left(\frac{1-a^{n-m}}{1-a}\right) p^{\mathtt{E}}(u_1, u_0) \in \frac{\rho}{2} U^+$ for any $m, n \geq n_1$, therefore $\tau - \frac{a^{n-m}}{1-a} p^{\mathtt{E}}(u_1, u_0) - \frac{\rho}{2} U^+ \subseteq \tau - \rho U^+ \subseteq \mathtt{E}^+$,

$$p^{\mathtt{E}}(u_n, u_m) \preceq a^m \left(\frac{1 - a^{n-m}}{1-a}\right) p^{\mathtt{E}}(u_1, u_0) \ll \tau \text{ for all } n, m \geq n_1$$

Hence (u_n) is an e-Cauchy sequence, since u is e-complete $\exists\ u \in \mathtt{X}$ such that $u_n \xrightarrow{e} u$. For a given $\tau \ggg 0_E$, choose $n_2 \in \mathbb{N}$, such that $p^{\mathtt{E}}(u, u_n) \ll (1-a)\frac{\tau}{m}$ for all $n \geq n_2$. Consider for all $n \geq n_2$

$$\begin{aligned} p^{\mathtt{E}}(u, \mathtt{G}u) &\preceq p^{\mathtt{E}}(u, u_n) + p^{\mathtt{E}}(u_n, \mathtt{G}u) \\ &= p^{\mathtt{E}}(u, u_n) + p^{\mathtt{E}}(Tu_{n-1}, \mathtt{G}u) \\ &\preceq p^{\mathtt{E}}(u, u_n) + ap^{\mathtt{E}}(u_{n-1}, u) \\ &\ll (1-a)\frac{\tau}{m} + a\frac{\tau}{m} = \frac{\tau}{m} \end{aligned}$$

Therefore $p^{\mathtt{E}}(u, \mathtt{G}u) \ll \frac{\tau}{m}$ for any $\frac{\tau}{m} \ggg 0_{\mathtt{E}}$ and $m \in \mathbb{N}$, therefore $\frac{\tau}{m} - p^{\mathtt{E}}(u, \mathtt{G}u) \in E^+$ for all $m \in \mathbb{N}$, which implies $-p^{\mathtt{E}}(u, \mathtt{G}u) \in E^+$, but $p^{\mathtt{E}}(u, \mathtt{G}u) \in E^+$, therefore

$$p^{\mathtt{E}}(u, \mathtt{G}u) = 0_E$$

Hence $u = \mathtt{G}u$. Let $v \in \mathtt{X}$ be such that $u = \mathtt{G}u$ and $v = \mathtt{G}v$, then consider

$$p^{\mathtt{E}}(u, v) = p^{\mathtt{E}}(\mathtt{G}u, \mathtt{G}v) \preceq ap^{\mathtt{E}}(u, v),$$

$p^{\mathtt{E}}(u, v) = 0_{\mathtt{E}}$
Hence $u = v$. □

Theorem 2.2. *Let* $(\mathtt{X}, p^{\mathtt{E}})$ *be an* $e-$ *complete* $\mathtt{E}-$ *metric space with closed positive cone* $\mathtt{E}^+$ *such that* $(\mathtt{E}^+)^{\ominus} \neq \phi$. *A mapping* $\mathtt{G} : \mathtt{X} \to \mathtt{X}$ *and there exist a non-negative number* b_1, b_2, b_3 *satisfying* $b_1 + b_2 + b_3 < 1$ *such that for each* $u, v \in \mathtt{X}$,

$$p^{\mathtt{E}}(\mathtt{G}u, \mathtt{G}v) \preceq b_1 p^{\mathtt{E}}(u, \mathtt{G}v) + b_2 p^{\mathtt{E}}(v, \mathtt{G}u) + b_3 p^{\mathtt{E}}(u, v) \tag{2.4}$$

Then $\mathtt{G}$ *has a unique fixed point in* $\mathtt{X}$.

Proof. Consider $u_0 \in \mathtt{X}, u_{n+1} = \mathtt{G}u_n = \mathtt{G}^{\mathtt{n}} u_0 \forall n \in \mathbb{N}$. For any $n \in \mathbb{N}, u_n \neq u_{n+1}$

$$\begin{aligned} p^{\mathtt{E}}(u_n, u_{n+1}) &= p^{\mathtt{E}}(\mathtt{G}u_n, \mathtt{G}u_{n-1}) \\ &\preceq b_1 p^{\mathtt{E}}(u_n, \mathtt{G}u_{n-1}) + b_2 p^{\mathtt{E}}(u_{n-1}, \mathtt{G}u_n) + b_3 p^{\mathtt{E}}(u_n, u_{n-1}) \\ &= b_1 p^{\mathtt{E}}(u_n, u_n) + b_2 p^{\mathtt{E}}(u_{n-1}, u_{n+1}) + b_3 p^{\mathtt{E}}(u_n, u_{n-1}) \\ &\preceq b_2 p^{\mathtt{E}}(u_{n-1}, u_n) + b_2 p^{\mathtt{E}}(u_n, u_{n+1}) + b_3 p^{\mathtt{E}}(u_n, u_{n-1}) \\ (1-b_2) p^{\mathtt{E}}(u_n, u_{n+1}) &\preceq (b_2 + b_3) p^{\mathtt{E}}(u_{n-1}, u_n) \end{aligned}$$

Therfore,

$$p^{\mathtt{E}}(u_n, u_{n+1}) \preceq \left(\frac{b_2 + b_3}{1 - b_2}\right) p^{\mathtt{E}}(u_{n-1}, u_n) \tag{2.5}$$

Here $s = \dfrac{b_2 + b_3}{1 - b_2} < 1$ and From equation (2.5) Continuing the above process, we get

$$p^{\mathtt{E}}(u_n, u_{n+1}) \preceq s^n p^{\mathtt{E}}(u_0, u_1) \tag{2.6}$$

For $n > m$,

$$\begin{aligned} p^{\mathtt{E}}(u_m, u_n) &\preceq p^{\mathtt{E}}(u_m, u_{m+1}) + p^{\mathtt{E}}(u_{m+1}, u_{m+2}) + \cdots + p^{\mathtt{E}}(u_{n-1}, u_n) \\ &\preceq \left(s^m + s^{m+1} + \cdots + s^{n-1}\right) p^{\mathtt{E}}(u_1, u_0) \\ &\preceq s^m \left(1 + s + s^2 + \cdots + s^{n-m-1}\right) p^{\mathtt{E}}(u_1, u_0) \\ &\preceq s^m \left(\frac{1 - s^{n-m}}{1 - s}\right) p^{\mathtt{E}}(u_1, u_0) \end{aligned}$$

for $n > m$, consider

$$\begin{aligned} p^{\mathtt{E}}(u_m, u_n) &\preceq p^{\mathtt{E}}(u_m, u_{m+1}) + p^{\mathtt{E}}(u_{m+1}, u_{m+2}) + \cdots + p^{\mathtt{E}}(u_{n-1}, u_n) \\ &\preceq \left(s^m + s^{m+1} + \cdots + s^{n-1}\right) p^{\mathtt{E}}(u_1, u_0) \\ &\preceq s^m \left(1 + s + s^2 + \cdots + s^{n-m-1}\right) p^{\mathtt{E}}(u_1, u_0) \\ &\preceq s^m \left(\frac{1 - s^{n-m}}{1-s}\right) p^{\mathtt{E}}(u_1, u_0) \end{aligned}$$

For any $\tau \ggg 0_{\mathtt{E}}, \exists \rho > 0$ such that $\tau - \rho U^+ \subseteq \mathtt{E}^+$ and $n_3 \in \mathbb{N} \ni s^m \left(\frac{1-s^{n-m}}{1-s}\right) p^{\mathtt{E}}(u_1, u_0) \in \frac{\rho}{2} U^+$ for any $m, n \geq n_3$, therefore $\tau - \frac{r^{n-m}}{1-r} p^{\mathtt{E}}(u_1, u_0) - \frac{\rho}{2} U^+ \subseteq \tau - \rho U^+ \subseteq \mathtt{E}^+$, hence

$$p^{\mathtt{E}}(u_n, u_m) \preceq s^m \left(\frac{1 - s^{n-m}}{1-s}\right) p^{\mathtt{E}}(u_1, u_0) \ll \tau \text{ for all } n, m \geq n_3$$

Hence (u_n) is an e-Cauchy sequence, since u is e-complete $\exists\ u \in \mathtt{X}$ such that $u_n \xrightarrow{e} u$. Given $\tau \ggg 0_E$, choose $n_4 \in \mathbb{N}$, such that $p^{\mathtt{E}}(u, u_n) \ll (1-s)\frac{\tau}{m}\ \forall\ n \geq n_4$.

$$\begin{aligned} p^{\mathtt{E}}(u, \mathtt{G}u) &\preceq p^{\mathtt{E}}(u, u_n) + p^{\mathtt{E}}(u_n, \mathtt{G}u) \\ &= p^{\mathtt{E}}(u, u_n) + p^{\mathtt{E}}(T u_{n-1}, \mathtt{G}u) \\ &\preceq p^{\mathtt{E}}(u, u_n) + r p^{\mathtt{E}}(u_{n-1}, u) \\ &\ll (1-s)\frac{\tau}{m} + s\frac{\tau}{m} = \frac{\tau}{m} \end{aligned}$$

Therefore $p^{\mathtt{E}}(u, \mathtt{G}u) \ll \frac{\tau}{m},\ \forall\ \frac{\tau}{m} \ggg 0_{\mathtt{E}}$ and $m \in \mathbb{N}$, $\frac{\tau}{m} - p^{\mathtt{E}}(u, \mathtt{G}u) \in E^+$ for all $m \in \mathbb{N}$, which implies $-p^{\mathtt{E}}(u, \mathtt{G}u) \in E^+$, but $p^{\mathtt{E}}(u, \mathtt{G}u) \in E^+$,

$$p^{\mathtt{E}}(u, \mathtt{G}u) = 0_E$$

Hence $u = \mathtt{G}u$. Let $v \in \mathtt{X}$ be such that $u = \mathtt{G}u$ and $v = \mathtt{G}v$, then consider

$$p^{\mathtt{E}}(u, v) = p^{\mathtt{E}}(\mathtt{G}u, \mathtt{G}v) \preceq s p^{\mathtt{E}}(u, v),$$

$p^{\mathtt{E}}(u, v) = 0_{\mathtt{E}}$

Hence $u = v$. □

Corollary 2.3. *Let $(\mathtt{X}, p^{\mathtt{E}})$ be an $e-$ complete $\mathtt{E}-$ Metric space with closed positive cone $\mathtt{E}^+$ such that $(\mathtt{E}^+)^{\ominus} \neq \phi$. A self mapping $\mathtt{G}$ of $\mathtt{X}$ satisfies*

$$p^{\mathtt{E}}(\mathtt{G}u, \mathtt{G}v) \preceq a \max\left\{p^{\mathtt{E}}(u, v), p^{\mathtt{E}}(u, \mathtt{G}u), p^{\mathtt{E}}(v, \mathtt{G}v), p^{\mathtt{E}}(u, \mathtt{G}v), p^{\mathtt{E}}(v, \mathtt{G}u)\right\}. \tag{2.7}$$

For all $u, v \in \mathtt{X}$ & $a \in (0, 1)$. Then $\mathtt{G}$ has a unique fixed point in $\mathtt{X}$.

Corollary 2.4. *Let $(\mathtt{X}, p^{\mathtt{E}})$ be an $e-$ complete $\mathtt{E}-$ Metric space with closed positive cone $\mathtt{E}^+$ such that $(\mathtt{E}^+)^{\ominus} \neq \phi$. A self mapping $\mathtt{G}$ of $\mathtt{X}$ satisfies*

$$p^{\mathtt{E}}(\mathtt{G}u, \mathtt{G}v) \preceq a \max\left\{p^{\mathtt{E}}(u, v), p^{\mathtt{E}}(u, \mathtt{G}u), p^{\mathtt{E}}(v, \mathtt{G}v), \frac{p^{\mathtt{E}}(u, \mathtt{G}v) + p^{\mathtt{E}}(v, \mathtt{G}u)}{2}\right\}. \tag{2.8}$$

For all $u, v \in \mathtt{X}$ & $a \in (0, 1)$. Then $\mathtt{G}$ has a unique fixed point in $\mathtt{X}$.

Definition 2.1. Let Ψ be the class of all real valued continuous functions $\psi : (\mathbb{R}^+)^6 \to \mathbb{R}^+$ is non-decreasing. Then it satisfies

For any $u, v \in \mathbb{R}^+$,

If it is either $u \preceq \psi\left(v, u, 0, v + u, v, \frac{u+v}{2}\right)$ or $u \preceq \psi\left(v, v, u, v, u, v, \frac{u}{2}\right)$ or $u \preceq \psi\left(u, v, u, u, v, v\right)$
$\exists$ a real number $0 < a < 1$ such that $u \preceq av$

Theorem 2.5. *Let $(\mathtt{X}, p^{\mathtt{E}})$ be an $e-$ complete $\mathtt{E}-$ metric space with closed positive cone such that $(\mathtt{E}^+)^{\ominus} \neq \phi$. A continuous self mapping $\mathtt{G}$ of $\mathtt{X}$ satisfies*

$$p^{\mathtt{E}}(\mathtt{G}u, \mathtt{G}v) \preceq \psi\left(p^{\mathtt{E}}(u,v), p^{\mathtt{E}}(u,\mathtt{G}u), p^{\mathtt{E}}(u,\mathtt{G}v), p^{\mathtt{E}}(v,\mathtt{G}u), p^{\mathtt{E}}(v,\mathtt{G}v), \frac{p^{\mathtt{E}}(u,\mathtt{G}u)+p^{\mathtt{E}}(v,\mathtt{G}v)}{2}\right). \tag{2.9}$$

$\forall u, v \in \mathtt{X}$. Then $\mathtt{G}$ has a unique fixed point in $\mathtt{X}$.

Proof. For any $u_0 \in \mathtt{X}$, and $n \geq 1$ such that $u_1 = \mathtt{G}u_0$ Continuing this process upto $(n+1)$ terms, we get $u_{n+1} = \mathtt{G}u_n = \mathtt{G}^{\mathtt{n}}u_0$
Consider

$$\begin{aligned}
p^{\mathtt{E}}(u_n, u_{n+1}) &= p^{\mathtt{E}}(\mathtt{G}u_n, \mathtt{G}u_{n-1}) \\
&\preceq \psi\left(p^{\mathtt{E}}(u_n, u_{n-1}), p^{\mathtt{E}}(u_n, \mathtt{G}u_n), p^{\mathtt{E}}(u_n, \mathtt{G}u_{n-1}), p^{\mathtt{E}}(u_{n-1}, \mathtt{G}u_n), p^{\mathtt{E}}(u_{n-1}, \mathtt{G}u_{n-1}), \frac{p^{\mathtt{E}}(u_n, \mathtt{G}u_n)+p^{\mathtt{E}}(u_{n-1}, \mathtt{G}u_{n-1})}{2}\right) \\
&= \psi\left(p^{\mathtt{E}}(u_n, u_{n-1}), p^{\mathtt{E}}(u_n, u_{n+1}), p^{\mathtt{E}}(u_n, u_n), p^{\mathtt{E}}(u_{n-1}, u_{n+1}), p^{\mathtt{E}}(u_{n-1}, u_n), \frac{p^{\mathtt{E}}(u_n, u_{n+1})+p^{\mathtt{E}}(u_{n-1}, u_n)}{2}\right) \\
&\preceq \psi\left(p^{\mathtt{E}}(u_n, u_{n-1}), p^{\mathtt{E}}(u_n, u_{n+1}), 0, (p^{\mathtt{E}}(u_{n-1}, u_n)+p^{\mathtt{E}}(u_n, u_{n+1})), p^{\mathtt{E}}(u_{n-1}, u_n), \frac{p^{\mathtt{E}}(u_n, u_{n+1})+p^{\mathtt{E}}(u_{n-1}, u_n)}{2}\right)
\end{aligned}$$

From definition of 2.1

$$p^{\mathtt{E}}(u_n, u_{n+1}) \preceq a p^{\mathtt{E}}(u_n, u_{n-1})$$

Similarly,

$$p^{\mathtt{E}}(u_n, u_{n-1}) \preceq a p^{\mathtt{E}}(u_{n-1}, u_{n-2})$$

Then

$$p^{\mathtt{E}}(u_n, u_{n+1}) \preceq a p^{\mathtt{E}}(u_n, u_{n-1}) \preceq a^2 p^{\mathtt{E}}(u_{n-1}, u_{n-2})$$

On Continuing this inequality, we get

$$p^{\mathtt{E}}(u_n, u_{n+1}) \preceq a^n p^{\mathtt{E}}(u_0, u_1)$$

Next for $n > m$,

$$\begin{aligned}
p^{\mathtt{E}}(u_m, u_n) &\preceq p^{\mathtt{E}}(u_m, u_{m+1}) + p^{\mathtt{E}}(u_{m+1}, u_{m+2}) + \cdots + p^{\mathtt{E}}(u_{n-1}, u_n) \\
&\preceq \left(\sum_{k=m}^{n-1} a^k\right) p^{\mathtt{E}}(u_1, u_0) \\
&\preceq a^m \left(\sum_{k=0}^{n-m-1} a^k\right) p^{\mathtt{E}}(u_1, u_0) \\
&\preceq a^m \left(\frac{1-a^{n-m}}{1-a}\right) p^{\mathtt{E}}(u_1, u_0) \\
&= a^m M l
\end{aligned}$$

Where $M = \left(\frac{1-a^{n-m}}{1-a}\right)$ and $l = p^{\mathtt{E}}(u_1, u_0)$. Let $\tau \ggg 0_{\mathtt{E}}$ be given, choose $\rho > 0 \ni \tau - \rho U^+ \subseteq \mathtt{E}^+$ and $n_5 \in \mathbb{N}$ such that $a^m \left(\frac{1-a^{n-m}}{1-a}\right) p^{\mathtt{E}}(u_1, u_0) \in \frac{\rho}{2} U^+$ for any $m, n \geq n_5$, $\tau - \frac{a^{n-m}}{1-a} p^{\mathtt{E}}(u_1, u_0) - \frac{\rho}{2} U^+ \subseteq \tau - \rho U^+ \subseteq \mathtt{E}^+$,

$$p^{\mathtt{E}}(u_n, u_m) \preceq a^m \left(\frac{1-a^{n-m}}{1-a}\right) p^{\mathtt{E}}(u_1, u_0) \ll \tau \ \forall\ n, m \geq n_6$$

Hence (u_n) is an e-Cauchy sequence, since u is e-complete $\exists\ u \in \mathtt{X}$ such that $u_n \xrightarrow{e} u$. For a given $\tau \ggg 0_E$, choose $n_6 \in \mathbb{N}$, such that $p^{\mathtt{E}}(u, u_n) \ll (1-a)\frac{\tau}{m}\ \forall\ n \geq n_6 \ni$

$$\begin{aligned}
p^{\mathtt{E}}(u, \mathtt{G}u) &\preceq p^{\mathtt{E}}(u, u_{n+1}) + p^{\mathtt{E}}(u_{n+1}, \mathtt{G}u) \\
&= p^{\mathtt{E}}(u, u_{n+1}) + p^{\mathtt{E}}(\mathtt{G}u_n, \mathtt{G}u) \\
&\preceq p^{\mathtt{E}}(u, u_{n+1}) + \psi\left(p^{\mathtt{E}}(u_n, u), p^{\mathtt{E}}(u_n, \mathtt{G}u_n), p^{\mathtt{E}}(u_n, \mathtt{G}u), p^{\mathtt{E}}(u, \mathtt{G}u_n), p^{\mathtt{E}}(u, \mathtt{G}u), \frac{p^{\mathtt{E}}(u_n, \mathtt{G}u_n) + p^{\mathtt{E}}(u, \mathtt{G}u)}{2}\right) \\
&= p^{\mathtt{E}}(u, u_{n+1}) + \psi\left(p^{\mathtt{E}}(u_n, u), p^{\mathtt{E}}(u_n, u_{n+1}), p^{\mathtt{E}}(u_n, \mathtt{G}u), p^{\mathtt{E}}(u, u_{n+1}), p^{\mathtt{E}}(u, \mathtt{G}u), \frac{p^{\mathtt{E}}(u_n, u_{n+1}) + p^{\mathtt{E}}(u, \mathtt{G}u)}{2}\right)
\end{aligned}$$

Taking $u_n \xrightarrow{e} u$, we get

$$p^{\mathtt{E}}(u, \mathtt{G}u) \preceq 0 + \psi\left(0, 0, p^{\mathtt{E}}(u, \mathtt{G}u), 0, p^{\mathtt{E}}(u, \mathtt{G}u), \frac{p^{\mathtt{E}}(u, \mathtt{G}u)}{2}\right)$$

From the definition 2.1, we have

$$p^{\mathtt{E}}(u, \mathtt{G}u) \preceq 0_{\mathtt{E}}.$$

Here, $p^{\mathtt{E}}(u, \mathtt{G}u) \notin \mathtt{E}^+$.

$\therefore$ *Only possible* $p^{\mathtt{E}}(u, \mathtt{G}u) = 0$. Hence, $\mathtt{G}u = u$.

Suppose $\mathtt{G}$ has another fixed point say $v = \mathtt{G}v$, then

$$\begin{aligned}
p^{\mathtt{E}}(u, v) &= p^{\mathtt{E}}(\mathtt{G}u, \mathtt{G}v) \\
&\preceq \psi\left(p^{\mathtt{E}}(u, v), p^{\mathtt{E}}(u, \mathtt{G}u), p^{\mathtt{E}}(u, \mathtt{G}v), p^{\mathtt{E}}(v, \mathtt{G}u), p^{\mathtt{E}}(v, \mathtt{G}v), \frac{p^{\mathtt{E}}(u, \mathtt{G}u) + p^{\mathtt{E}}(v, \mathtt{G}v)}{2}\right) \\
&= \psi\left(p^{\mathtt{E}}(u, v), 0, p^{\mathtt{E}}(u, v), p^{\mathtt{E}}(u, v), 0, 0\right)
\end{aligned}$$

From the definition of 2.1, we get

$p^{\mathtt{E}}(u, v) \preceq 0_{\mathtt{E}}, -p^{\mathtt{E}}(u, v) \notin \mathtt{E}^+$
$\therefore\ p^{\mathtt{E}}(u, v) = 0_{\mathtt{E}}$

Hence $u = v$. □

3 Conclusions

In the article that is being presented, we describe fixed point results with various types of contraction in an $\mathtt{E}-$ metric space. We also established a new implicit relation, which enables the contraction to reach a fixed point result more conveniently. The main result of my article can be extended to random cone metric space and integral type fixed point results.

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Competing Interests

Authors have declared that no competing interests exist.

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