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Constancy of Holomorphic Sectional Curvature for Indefinite $S-$Manifolds

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Research Article

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Abstract

Kumar et al. (2009) provided an algebraic characterization for an indefinite Sasakian manifold to reduce to a space of constant $\phi-$holomorphic sectional curvature. In this present paper, we generalize the characterization for indfinite $S-$manifolds, which is a space of constant $\phi-$holomorphic sectional curvature if and only if $R(X,\phi X)X$ *is proportional to* ϕX.

Keywords: sectional curvature; $S-$*manifold; indefinite* $S-$*manifold;*

1 Introduction

For an almost Hermitian manifold (M^{2n}, g, J) with $dim(M) = 2n > 4$, Tanno (1973) has proved;

Theorem 1.1

Let $dim(M) = 2n > 4$ *and assume that almost Hermitian manifold* (M^{2n}, g, J) *satisfies*

$$R(JX, JY, JZ, JX) = R(X, Y, Z, X) \quad (1)$$

for every tangent vectors X, Y and Z. Then (M^{2n}, g, J) is of constant holomorphic sectional curvature at x if and only if

$$R(X, JX)X \text{ is proportional to } JX \quad (2)$$

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for every tangent vector X at $x \in M$.

Tanno (1973) has also proved an analogous theorem for Sasakian manifolds as

Theorem 1.2

A Sasakian manifold ≥ 5 *is of constant* $\phi-$ *sectional curvature if and only if*

$$R(X,\phi X)X \ \ is \ \ proportional \ \ to \ \ \phi X \tag{3}$$

for every tangent vector X such that $g(X,\xi)=0$.

Nagaich (1993) has proved the generalized version of Theorem 1.1, for indefinite almost Hermitian manfiolds as

Theorem 1.3

Let (M^{2n},g,J), $(n>2)$ *be an indefinite almost Hermitian manifold satisfies (1), then* (M^{2n},g,J) *is of constant holomorphic sectional curvature at* x *if and only if*

$$R(X,JX)Xisproportionalto JX \tag{4}$$

for every tangent vector X at $x \in M$.

Kumar et al. (2009) generalized Theorem 1.2 for an indefinite Sasakian manifold as

Theorem 1.4

Let $(M^{2n+1},\phi,\eta,\xi,g)$, $(n \geq 2)$ *be an indefinit Sasakian manifold. Then* M^{2n+1} *is of constant* $\phi-$ *sectional curvature if and only if*

$$R(X,\phi X)X \ \ is \ \ proportional \ \ to \ \ \phi X \tag{5}$$

for every vector field X such that $g(X,\xi)=0$.

In this paper, we generalize Theorem 1.4 for an indefinite $S-$manifold by proving the following

Theorem 1.5

Let $(\overline{M}^{2n+r}, \overline{\phi}, \overline{\eta}^{\alpha}, \overline{\xi}_{\alpha}, \overline{g})$, $(n \geq 2)$ *be an indefinite* $S-$*manifold. Then* M^{2n+r} *is of constant* $\phi-$*sectional curvature if and only if*

$$R(X, \overline{\phi}X)X \text{ is proportional to } \overline{\phi}X \tag{6}$$

for every vector field X such that $g(X, \overline{\xi}_{\alpha}) = 0$, for any $\alpha \in \{1, \cdots, r\}$.

2 Preliminaries

2.1 An $S-$manifold

Definition 2.1

A $g.f.f-$*manfiold* $(\overline{M}^{2n+r}, \overline{\phi}, \overline{\xi}_{\alpha}, \overline{\eta}^{\alpha})$ *is called an* $S-$*manifold if it is normal and* $d\overline{\eta}^{\alpha} = \Phi$, *for any* $\alpha \in \{1, \cdots, r\}$, *where* $\Phi(X, Y) = \overline{g}(X, \phi Y)$ *for any* X, $Y \in \Gamma(T\overline{M})$ *The normality condition is expressed by the vanishing of the tensor field* $N = N_{\phi} + \sum_{\alpha=1}^{r} 2d\overline{\eta}^{\alpha} \otimes \overline{\xi}_{\alpha}$, N_{ϕ} *being the Nijenhuis torsion of* ϕ.

If $(\overline{M}^{2n+r}, \overline{\phi}, \overline{\xi}_{\alpha}, \overline{\eta}^{\alpha})$ is an $S-$manifold, then it is known (Blair, 1970) that

$$(\overline{\nabla}_X \overline{\phi})Y = \overline{g}(\overline{\phi}X, \overline{\phi}Y)\overline{\xi} + \overline{\eta}(Y)\overline{\phi}^2(X) \tag{7}$$

$$\overline{\nabla}_X \overline{\xi}_{\alpha} = -\overline{\phi}X, \tag{8}$$

where $\overline{\xi} = \sum_{\alpha=1}^{r} \overline{\xi}_{\alpha}$ and $\overline{\eta} = \sum_{\alpha=1}^{r} \overline{\eta}^{\alpha}$.

Theorem 2.1

[1] An $S-$*manifold* M^{2n+r} *has constant* $\phi-$*sectional curvature* c *if and only if its curvature tensor field satisfies*

$$\overline{R}(X, Y, Z)$$

$$
\begin{aligned}
&= \frac{(c+3r)}{4}\{\overline{g}(\overline{\phi}X,\overline{\phi}Z)\overline{\phi}^2Y - \overline{g}(\overline{\phi}Y,\overline{\phi}Z)\overline{\phi}^2X\} \\
&\quad + \frac{(c-r)}{4}\{\Phi(Z,Y)\phi X - \Phi(Z,X)\phi Y + 2\Phi(X,Y)\phi Z\} \\
&\quad - \overline{g}(\overline{\phi}Z,\overline{\phi}Y)\overline{\eta}(X)\overline{\eta} + \overline{g}(\overline{\phi}Z,\overline{\phi}X)\overline{\eta}(Y)\overline{\eta} \\
&\quad - \overline{\eta}(Y)\overline{\eta}(Z)\overline{\phi}^2X + \overline{\eta}(Z)\overline{\eta}(X)\overline{\phi}^2Y
\end{aligned} \tag{9}
$$

for any vector fields $X,Y,Z,W \in \Gamma(T\overline{M})$.

An $S-$manifold M^{2n+r} with constant $\phi-$sectional curvature c is called an $S-$*space form* and denoted by $M^{2n+r}(c)$.

When $r=1$, an $S-$space form $M^{2n+1}(c)$ reduces to a Sasakian space form (Blair, 2002) and (9) reduces to

$$
\begin{aligned}
&4\overline{R}(X,Y,Z) \\
&= (c+3)\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y\} \\
&\quad + (c-1)\{\Phi(X,Z)\phi Y - \Phi(Y,Z)\phi X + 2\Phi(X,Y)\phi Z \\
&\quad - \overline{g}(Z,Y)\eta(X)\eta + \overline{g}(Z,X)\eta(Y)\eta \\
&\quad - \eta(Y)\eta(Z)X + \eta(Z)\eta(X)Y\}
\end{aligned} \tag{10}
$$

for any vector fields $X,Y,Z,W \in \Gamma(T\overline{M})$, where $\xi_1 = \xi$ and $\eta^1 = \eta$.

When $r=0$, an $S-$space form $M^{2n}(c)$ becomes a complex space form and (9) moves to

$$
\begin{aligned}
4R(X,Y,Z) = c\{&\overline{g}(Y,Z)X - \overline{g}(X,Z)Y \\
&+ \Phi(X,Z)\phi Y - \Phi(Y,Z)\phi X + 2\Phi(X,Y)\phi Z\}
\end{aligned} \tag{11}
$$

3 Proof of Theorem 1.5

Let $(\overline{M}^{2n+r},\overline{\phi},\overline{\xi}_\alpha,\overline{\eta}^\alpha)$, $\alpha \in \{1,\cdots,r\}$, $(n \geq 2)$ be an indefinite $S-$manifold. To prove the theorem for $n \geq 2$, we shall consider cases when $n=2$ and when $n>2$, that is, when $n \geq 3$.

Case I

When the metric is space-like, that is, when $\overline{g}(X,X) = \overline{g}(Y,Y)$. The proof is similar as given by Lee and Jin (2011), so we drop the proof.

Case II

When the metric is time-like, that is, when $\overline{g}(X,X) = -\overline{g}(Y,Y)$. Here, if X is space-like, then Y is time-like or vice versa. First of all, assume that $\overline{M}$ is of constant $\phi-$holomorphic sectional curvature. Then (9) gives

$$R(X,\phi X)X = c\phi X \tag{12}$$

Conversely, let $\{X,Y\}$ be an orthonormal pair of tangent vectors such that $\overline{g}(\phi X,Y) = \overline{g}(X,Y) = \overline{g}(Y,\overline{\xi}_\alpha) = 0$, $\alpha \in \{1,\cdots,r\}$ and $n \geq 3$. Then $\ddot{X} = \dfrac{X+\overline{\phi}Y}{\sqrt{2}}$ and $\ddot{Y} = \dfrac{\overline{\phi}^2 X + \overline{\phi}Y}{\sqrt{2}}$ also form an orthonormal pair of tangent vectors such that $\overline{g}(\phi\ddot{X},\ddot{Y}) = 0$.

Then (12) and curvature properties give

$$\begin{aligned} 0 = R(\ddot{X},\phi\ddot{X},\ddot{Y},\ddot{X}) &= \overline{g}(R(X,\phi X,X),\phi X) - \overline{g}(R(Y,\phi Y,Y),\phi Y) \\ &\quad - 2\overline{g}(R(X,\phi Y,Y),\phi Y) + 2\overline{g}(R(X,\phi X,Y),\phi X) \end{aligned} \tag{13}$$

From the assumption, we see that the last two terms of the right hand side vanish. Therefore, we get $c(X) = c(Y)$.

Now, if $sp\{U,V\}$ is $\phi-$holomorphic, then for $\phi U = aU + bV$ where a and b are constant.

Then we have

$$sp\{U,\phi U\} = sp\{U, aU + bV\} = sp\{U,V\}$$

Similarly,

$$sp\{V,\phi V\} = sp\{U,V\}, \qquad sp\{U,\phi U\} = sp\{V,\phi V\}$$

These imply

$$R(U,\phi U,U,\phi U) = R(V,\phi V,V,\phi V), or \quad c(U) = c(V)$$

If $\{U,V\}$ is not $\phi-$holomorphic section, then we can choose unit vectors $X \in sp\{U,\phi U\}^\perp$ and $Y \in sp\{V,\phi V\}^\perp$ such that $sp\{X,Y\}$ is $\phi-$holomorphic. Thus we get

$$c(U) = c(X) = c(Y) = c(V),$$

which shows that any $\phi-$holomorphic section has the same $\phi-$ holomorphic sectional curvature.

Now, let $n=2$ and let $\{X,Y\}$ be a set of orthonormal vectors such that $\overline{g}(X,X)=-\overline{g}(Y,Y)$ and $\overline{g}(X,\phi X)=0$, we have $c(X)=c(Y)$ as before. Using the property (6), we get

$$
\begin{aligned}
&R(X,\phi X,X)=c(X)\phi X\\
&R(X,\phi X,Y)=-R(X,\phi X,Y,\phi Y)\phi Y\\
&R(X,\phi Y,X)=-R(X,\phi Y,X,Y)Y-R(X,\phi Y,X,\phi Y)\phi Y\\
&R(X,\phi Y,Y)=R(X,\phi Y,Y,X)X+R(X,\phi Y,Y,\phi X)\phi X\\
&R(Y,\phi X,Y)=R(Y,\phi X,Y,X)X+R(Y,\phi X,Y,\phi X)\phi X\\
&R(Y,\phi X,X)=-R(Y,\phi X,X,Y)Y-R(Y,\phi X,X,\phi Y)\phi Y\\
&R(Y,\phi Y,X)=R(Y,\phi Y,X,\phi X)\phi X\\
&R(Y,\phi Y,Y)=-c(Y)\phi Y=-c(X)\phi Y
\end{aligned}
$$

Now, define $\hat{X}=aX+bY$ such that $a^2-b^2=1$ and $a^2\neq b^2$. Using above relations, we get

$$R(\ddot{X},\phi\ddot{X},\ddot{X})=C_1X+C_2Y+C_3\phi X+C_4\phi Y$$

Note that C_1 and C_2 are not necessary for argument and we have

$$
\begin{aligned}
C_3&=a^3c(X)+ab^2C_5\\
C_4&=-b^3c(X)-a^2bC_5,
\end{aligned}
\tag{14}
$$

where $C_5=R(X,\phi X,Y,\phi Y)+R(X,\phi Y,X,\phi Y)+R(X,\phi Y,Y,\phi X)$.

On the other hand,

$$R(\hat{X},\phi\hat{X},\hat{X})=c(\hat{X})\phi\hat{X}=c(\hat{X})\{a\phi X+b\phi Y\}\tag{15}$$

Comparing (14) and (15), we get

$$a^2c(X)+b^2C_5=c(\hat{X})\tag{16}$$

$$-b^2c(X)-a^2C_5=c(\hat{X})\tag{17}$$

On Solving (16) and (17), we have

$$c(X)=c(\hat{X})$$

Similary, we can prove

$$c(Y) = c(\hat{Y})$$

Therefore, $\overline{M}$ has constant $\phi-$ holomorphic sectional curvature. W

Theorem 3.1 (Kumar et al., 2009)

Let $(M^{2n+1},\phi,\eta,\xi,g)$, $(n \geq 2)$ be an indefinit Sasakian manifold. Then M^{2n+1} is of constant $\phi-$ sectional curvature if and only if

$$R(X,\phi X)X is proport ional to \phi X \tag{18}$$

for every vector field X such that $g(X,\xi) = 0$.

Proof. When $r = 1$, an indefinite $\mathrm{S}-$space form $M^{2n+1}(c)$ reduces to a Sasakian space form. The proof follows from (10) and Theorem 1.5.

Remark 3.1

In this paper, we have considered the cases of space-like and time-like vectors only. However, we are still investigating the same results for lightlike vectors.

4 Conclusion

Here is the brief discussion over the main outcome of this article revealed in favor of algebraic characterization in indefinite $\mathrm{S}-$manifolds. In Section 2, let us given the general Sasakian manifolds, called indefinite $\mathrm{S}-$manifolds. After then, we introduced curvature tensors in an inde finite $\mathrm{S}-$manifold. By using Riemannian curvature tensor on indefinite $\mathrm{S}-$manifold, we generalized algebraic result from Kumar et al (2009) on the indefinite $\mathrm{S}-$manifold. Restrictin to dimensions on indefinite $\mathrm{S}-$manifolds, we could have old results on indefinite almost hermitian manifolds.·

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