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Optimal Homotopy Analysis Methods for Solving the Linear and Nonlinear Fokker-Planck Equations

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Abstract

In this paper, linear and nonlinear Fokker-Planck equations are solved by approximate method, namely Optimal homotopy analysis method (OHAM). The Optimal homotopy analysis method is a combination of the homotopy analysis method and optimizing the convergence control parameter by minimizing the square residual error. The mathematical calculation and graphics have been obtained. The results have been obtained by OHAM are matches with the exact results and other approximate method like, Adomian decomposition method (ADM), Variation iteration method (VIM) and Homotopy perturbation method (HPM).

Keywords: Optimal homotopy analysis method, convergence control parameter, fokker-Planck equations.

2010 Mathematics Subject Classification:35L02; 35L60

1 Introduction

Nonlinear phenomena, that appear in many areas of scientific fields such as solid state physics, plasma physics, fluid mechanics, population models and chemical kinetics, can be modelled by

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nonlinear differential equations. In many different fields of science and engineering, it is important to obtain exact or numerical solution of the nonlinear partial differential equations, but it is still quite problematic that's need new methods for finding the approximate solutions. It is well-known that nonlinear ordinary differential equations (ODEs) and partial differential equations (PDEs) for boundary-value problems are much more difficult to solve than linear ODEs and PDEs, especially by means of analytic methods. Traditionally, perturbation and asymptotic techniques are widely applied to obtain analytic approximations of nonlinear problems in science, finance and engineering. Unfortunately, perturbation and asymptotic techniques are too strongly dependent upon small/large physical parameters in general, and thus are often valid only for weakly nonlinear problems. Thus, it is necessary to develop some analytic approximation methods, which are independent of any small/large physical parameters at all and besides valid for strongly nonlinear problems. The homotopy analysis method (HAM)[1, 2, 3, 4] is based on the concept of homotopy, a fundamental concept in topology and differential geometry, which was first time introduced by Shijun Liao in his PhD thesis (1992)is independent of any small/large physical parameters. The HAM is a general analytic approach to get series solutions of various types of nonlinear equations, including algebraic equations, ordinary differential equations, partial differential equations, differential-integral equations, differential-difference equation, and coupled equations of them. Researches have introduce many other methods based on HAM like, Homotopy perturbation method (HPM)[5, 6, 7], Optimal homotopy analysis method (OHAM)[3, 8] Optimal homotopy asymptotic method [9, 10], Spectral homotopy analysis method and Homotopy analysis transform method [11] to get the approximate solution of the nonlinear PDE.

Inspired and motivated by the ongoing research in this area, we apply approximate method, namely Optimal Homotopy analysis method (OHAM) for solving the linear and nonlinear Fokker plank equation arising in the fluid flow through porous media and is matches with the exact results and with other approximate methods like, Adomian decomposition method [12, 13], Variation iteration method (VIM)[14] and Homotopy perturbation method (HPM)[5].

2 Fokker-Planck Equation

Using a microscope, Robert Brown (1773-1858) observed and documented the motion of large pollen grains suspended in water known as Brownian motion. The Fokker-Planck equation was first introduced by Fokker and Planck to describe the Brownian motion of particles [15]. A general Fokker-Planck equation can be derived from the Chapman-Kolmogorov equation. This equation has been used in different fields in natural sciences such as quantum optics, solid state physics, chemical physics, theoretical biology, circuit theory and fluid flow through porous media. The general form of Fokker-Plank equation is

$$\frac{\partial U}{\partial t} = \left[-\frac{\partial}{\partial x}A\left(x,t\right) + \frac{\partial^2}{\partial x^2}B\left(x,t\right)\right]U, \tag{2.1}$$

with initial condition $U\left(x,o\right) = f\left(x\right) \;\; ; \;\; x \in \mathbb{R}$. Equation (2.1) is also well known as a forward Kolmogorov equation. There exists another type of this equation is called a backward one as

$$\frac{\partial U}{\partial t} = \left[-A\left(x,t\right)\frac{\partial}{\partial x} + B\left(x,t\right)\frac{\partial^2}{\partial x^2}\right]U. \tag{2.2}$$

The nonlinear Fokker-Planck equation is a more general form of linear one which has also been applied in plasma physics, surface physics, astrophysics, the physics of polymer fluids and particle beams, nonlinear hydrodynamics, population dynamics, theory of electronic-circuitry and laser arrays, engineering, biophysics, human movement sciences, psychology and marketing. The nonlinear form of the Fokker-Planck equation can be expressed in the following way

$$\frac{\partial U}{\partial t} = \left[-\frac{\partial}{\partial x}A\left(x,t,U\right) + \frac{\partial^2}{\partial x^2}B\left(x,t,U\right)\right]U. \tag{2.3}$$

3 Basic Idea of OHAM

Consider the following nonlinear partial differential equation in a general form

$$\mathcal{N}\left[U\left(x,\ t\right)\right] \ = \ 0, \tag{3.1}$$

where, $\mathcal{N}$ is a nonlinear operator, x and t denote the independent variables and U is an unknown function. By means of the traditional HAM, we first construct the so-called zeroth order deformation equation as

$$\left(1-q\right)\mathcal{L}\left[\phi\left(x,t;q\right)-U_0\left(x,t\right)\right]=c_0\mathrm{q}\mathcal{N}\left[\phi\left(\mathrm{x},\mathrm{t};\mathrm{q}\right)\right], \tag{3.2}$$

where, $q\in[0,1]$ is the embedding parameter, c_0 is an auxiliary parameter also know as convergence control parameter, $\mathcal{L}$ is an auxiliary linear operator, $\phi\left(x,t;q\right)$ is an unknown function and $U_0(x,t)$ is an initial guess of $U(x,t)$. According to HAM we have great freedom to choose linear operator and initial guess. Obviously, when the embedding parameter $q=0$ and $q=1$, it holds $\phi\left(x,t;0\right)=U_0\left(x,t\right), \quad \phi\left(x,t;1\right)=U\left(x,t\right)$ respectively. Thus, as q increases from 0 to 1, the solution $\phi(x,t;q)$ varies from the initial guess $U_0(x,t)$ to the solution $U(x,t)$. Expanding $\phi(x,\ t;q)$ in Maclaurin series with respect to q, we have

$$\phi\left(x,t;q\right)=U_0\left(x,t\right)+\sum_{m=1}^{\infty}U_m\left(x,t\right)q^m, \tag{3.3}$$

where

$$U_m\left(x,t\right)=\frac{1}{m!}\frac{\partial^m\phi\left(x,t;q\right)}{\partial q^m}\bigg|_{q=0}.$$

Differentiating the equation (3.2) m-times with respect to q, dividing by $m!$ and finally setting $q=0$, we get the following mth-order deformation equation

$$\mathcal{L}\left[U_m\left(x,t\right)-\chi_mU_{m-1}\left(x,t\right)\right]=c_0\delta_m\left[U_{m-1}\left(x,t\right)\right], \tag{3.4}$$

where

$$\delta_m\left[U_{m-1}\left(x,t\right)\right]=\ \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}\left[\phi\left(x,t;q\right)\right]}{\partial q^{m-1}}\bigg|_{q=0}$$

and

$$\chi_m=\begin{cases}0, & m\leq 1\\ 1, & m>1\end{cases}$$

Applying inverse operator both side on equation (3.4)

$$U_m\left(x,t\right)=\chi_mU_{m-1}\left(x,t\right)+c_0\ \mathcal{L}^{-1}\left[\delta_m\left[U_{m-1}\left(x,t\right)\right]\right] \tag{3.5}$$

If the auxiliary linear operator [16], the initial guess and the convergence control parameter c_0 are properly chosen, then the series (3.3) converges at $q=1$, then we have

$$U\left(x,t\right)=U_0\left(x,t\right)+\sum_{m=1}^{\infty}U_m\left(x,t\right), \tag{3.6}$$

which must be one of the solution of original nonlinear equation (3.1), as proved by Liao [3]. The Convergence control parameter c_0 play an important role in the OHAM. One can gain convergent series solution simply by choosing a proper auxiliary parameter c_0. This is the reason why we call c_0 as the convergence-control parameter. In 2007, Yabushita et al. [17] applied the HAM to solve two coupled nonlinear ODEs. They suggested the so-called "optimization method" to find out the optimal convergence-control parameters by means of the minimum of the squared residual of governing equation as follow

$$E_m\left(c_0\right)=\iint_{\Omega}\left[\mathcal{N}\left\{\sum_{n=0}^{m}U_n\left(x,t\right)\right\}\right]^2d\Omega. \tag{3.7}$$

In real world problem the double integration of sum of square residual is very difficult, so we use it's approximate sum form as

$$E_m(c_0) = \frac{1}{(M+1)(N+1)} \sum_{i=0}^{M} \sum_{j=0}^{N} \left\{ \mathcal{N}\left[\sum_{n=0}^{m} U_n\left(\frac{i}{M}, \frac{j}{N} \right) \right]^2 \right\}. \tag{3.8}$$

Equation (3.8) gives the square residual error at m^{th}-order approximation. It is obvious that we can find the optimal value of convergence-control parameter c_0 at the any order of approximation. If there exists convergence-control parameter, c_0 for which we get the minimum of the squared residua E_m, is so called the optimal convergence-control parameter c_0.

We can prove the analytical convergence of OHAM solutions. To do this, let us state the following theorem which gives sufficient conditions for the convergence or divergence of the homotopy series.

Theorem 3.1. *Suppose that $A \subset R$ be a Banach space donated with the L^2 norm, over which the sequence $U_k(x,t)$ of the homotopy series $U(x,t;q) = \sum\limits_{k=1}^{\infty} U_k(x,t)\, q^k$ is defined for a prescribed value of c_0. Assume also that the initial approximation $U_0(x,t)$ remains inside the disc of the solution $U(x,t)$. Taking $r \in R$ be a constant, the following statements hold true:*

1. *If $\|U_{k+1}(x,t)\| \leq r\,\|U_k(x,t)\|$ for all k, given some $0 < r < 1$, then the series solution converges absolutely at $q = 1$ over the domain of definition of (x,t).*
2. *If $\|U_{k+1}(x,t)\| \geq r\,\|U_k(x,t)\|$ for all k, given some $r > 1$, then the series solution diverges at $q = 1$ over the domain of definition of (x,t).*

Proof

(1) Let $S_n(x,t)$ denote the sequence of partial sum of the homotopy series, we need to show that $S_n(x,t)$ is a Cauchy sequence in A. Also we have

$$\|S_{n+1}(x,t) - S_n(x,t)\| = \|U_{n+1}(x,t)\| \leq r\,\|U_n(x,t)\| \leq r^2\,\|U_{n-1}(x,t)\| \leq ... \leq r^{n+1}\,\|U_0(x,t)\|. \tag{3.9}$$

It should be remarked that owing to (3.9), all the approximations produced by the homotopy method will lie within the disk of $S_n(x,t)$. For every $m, n \in \mathbb{N}; n \geq m$, also using equation (3.9) and the triangle inequality, we have

$$\|S_n - S_m\| = \|(S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + ... + (S_{m+1} - S_m)\| \leq \frac{1 - r^{n-r}}{1 - r} r^{m+1}\,\|U_0\|. \tag{3.10}$$

Since $0 < r < 1$ we get from equation (3.10)

$$\lim_{n,m \to \infty} \|S_n(x,t) - S_m(x,t)\| = 0. \tag{3.11}$$

Therefore, $S_n(x,t)$ is a Cauchy sequence in the Banach space A, and we know that all Cauchy sequence in Banach space are convergent, that is the series solution is convergent.

The proof of (2) follows from the fact that under the hypothesis supplied in (2), suppose if possible there exist a number l such that, $l > r > 1$, so that the interval of convergence of the power series is $|\, q \,| < 1/l < 1$, which obviously contradiction with $q = 1$. $\boxdot$

4 Applications

We consider linear and nonlinear Fokker-Plank equations, which arias in many problem of fluid flow through porous media such as, Pressure head in unsaturated soil during infiltration phenomenon, Boussinesq's equation for infiltration phenomenon in unsaturated porous media and Advection-diffusion equation for concentration distribution in fluid flow through porous media.

4.1 Example of Linear Fokker-Planck equation

Consider the following linear Fokker-Planck equation

$$\mathrm{U}_t = \mathrm{U}_x + \mathrm{U}_{xx}, \tag{4.1}$$

subject to the initial condition

$$\mathrm{U}(x, 0) = x. \tag{4.2}$$

According to the OHAM, we choose the initial guess as, $U_0(x,t) = x$, and the auxiliary linear operator

$$\mathcal{L}[U(x,t;q)] = \frac{\partial U(x,t;q)}{\partial t} - \frac{\partial U(x,t;q)}{\partial x}, \tag{4.3}$$

The zeroth order deformation equation is

$$(1-q)\,\mathcal{L}[\phi(x,t;q) - U_0(x,t)] = c_0 \mathrm{q} \mathcal{N}[\phi(\mathrm{x},\mathrm{t};\mathrm{q})], \tag{4.4}$$

where the nonlinear operator can be define as

$$\mathcal{N}[\phi(x,t;q)] = \frac{\partial \phi(x,t;q)}{\partial t} - \frac{\partial \phi(x,t;q)}{\partial x} - \frac{\partial^2 \phi(x,t;q)}{\partial x^2}.$$

The corresponding m^{th}-order deformation equation is given by

$$\mathcal{L}[U_m(x,t) - \chi_m U_{m-1}(x,t)] = c_0 \delta_m [U_{m-1}(x,t)], \tag{4.5}$$

where

$$\delta_m(U_{m-1}) = (U_{m-1})_t - (U_{m-1})_x - (U_{m-1})_{xx}.$$

Applying the inverse operator on equation (4.5), we have

$$U_m(x,t) = \chi_m U_{m-1}(x,t) + c_0 \mathcal{L}^{-1}[\delta_m[U_{m-1}(x,t)]] \tag{4.6}$$

Solving equation (4.6), for $m = 1, 2, 3...$ We obtain

$$\begin{aligned} U_1(x,t) &= -c_0 t, \\ U_2(x,t) &= -c_0(1+c_0)\,t, \\ U_3(x,t) &= -c_0(1+c_0)^2\,t, \\ &\ \ ... \end{aligned} \tag{4.7}$$

By using Yabushita's approach we can find the optimal value of convergence control parameter by

$$E_m(c_0) = \frac{1}{(M+1)(N+1)} \sum_{i=0}^{M} \sum_{j=0}^{N} \left\{ \mathcal{N}\left[\sum_{n=0}^{m} U_n\left(\frac{i}{M}, \frac{j}{N}\right) \right]^2 \right\}. \tag{4.8}$$

We calculate equation (4.8) for $M = 20$ & $N = 20$ numbers of points and we get the optimal value of convergence control parameter $c_0 = -1.0000249174897184$ at minimum square residual error $E_5 = 6.51E - 35$ (Figure 1).

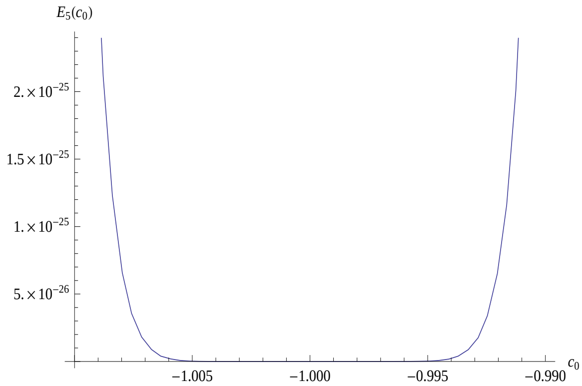

Figure 1: Square residual E_5 at 5^{th}-order approximation

By using this optimal value of c_0 we got the solution of (4.1) as

$$U(x,t)=\sum_{m=1}^{5}U_m(x,t)=x+t,$$

which is the exact solution and is same as obtained by ADM [13], VIM [14] and HPM [5].

4.2 Example of nonlinear Fokker-Planck equation

Consider the following nonlinear Fokker-Planck equation

$$\frac{\partial U}{\partial t}=\frac{\partial}{\partial x}\left(\frac{xU}{3}-\frac{4}{x}U^2\right)+\frac{\partial^2}{\partial x^2}\left(U^2\right), \tag{4.9}$$

with the initial condition

$$\mathrm{U}(x,0)=x^2. \tag{4.10}$$

According to the OHAM, we choose the initial guess as $U_0(x,t)=x^2$, and the auxiliary linear operator

$$\mathcal{L}[U(x,t;q)]=\frac{\partial U(x,t;q)}{\partial t}, \tag{4.11}$$

The zeroth order deformation equation is

$$(1-q)\,\mathcal{L}[\phi(x,t;q)-U_0(x,t)]=c_0\mathrm{q}\mathcal{N}[\phi(\mathrm{x},\mathrm{t};\mathrm{q})], \tag{4.12}$$

where the nonlinear operator can be define as

$$\mathcal{N}[\phi(x,t;q)]=\frac{\partial\phi(x,t;q)}{\partial t}-\frac{\partial}{\partial x}\left(\frac{x\phi(x,t;q)}{3}-\frac{4}{x}\phi^2(x,t;q)\right)-\frac{\partial^2\left(\phi^2(x,t;q)\right)}{\partial x^2}.$$

The corresponding m^{th}-order deformation equation is given by

$$\mathcal{L}[U_m(x,t)-\chi_m U_{m-1}(x,t)]=c_0\delta_m[U_{m-1}(x,t)], \tag{4.13}$$

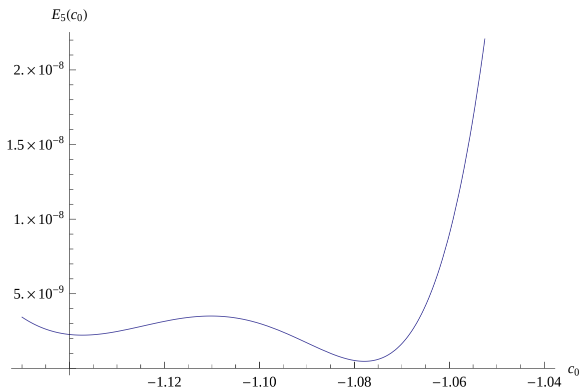

Figure 2: Square residual E_5 at 5^{th}-order approximation

where

$$\delta_m\left(U_{m-1}\right)=\left(U_{m-1}\right)_t-\left(\frac{x}{3}U_{m-1}-\frac{4}{x}\sum_{r=0}^{m-1}U_rU_{m-1-r}\right)_x-\left(\sum_{r=0}^{m-1}U_rU_{m-1-r}\right)_{xx}.$$

Applying the inverse operator on equation (4.13), we have

$$U_m\left(x,t\right)=\chi_mU_{m-1}\left(x,t\right)+c_0\,\mathcal{L}^{-1}\left[\delta_m\left[U_{m-1}\left(x,t\right)\right]\right] \tag{4.14}$$

Solving equation (4.14), for $m=1,2,3...$, we get

$$\begin{aligned} &U_1\left(x,t\right)=-c_0x^2t,\\ &U_2\left(x,t\right)=-c_0\left(1+c_0\right)x^2t+\tfrac{c_0^2x^2t^2}{2},\\ &U_3\left(x,t\right)=-c_0\left(1+c_0\right)^2x^2t+\tfrac{c_0^2(1+c_0)x^2t^2}{2}-\tfrac{c_0^3x^2t^3}{6},\\ &\ldots \end{aligned} \tag{4.15}$$

By using Yabushita's approach we can find the optimal value of convergence control parameter by

$$E_m\left(c_0\right)=\frac{1}{\left(M+1\right)\left(N+1\right)}\sum_{i=0}^{M}\sum_{j=0}^{N}\left\{\mathcal{N}\left[\sum_{n=0}^{m}U_n\left(\frac{i}{M},\frac{j}{N}\right)\right]^2\right\}. \tag{4.16}$$

We calculate equation (4.16) for $M=20$ & $N=20$ numbers of points and we get the optimal value of convergence control parameter $c_0=-1.0779113454061617$ at minimum square residual error $E_5=4.73E-10$. By using this optimal value of c_0 we got the solution of (4.9) as

$$\begin{aligned} U\left(x,t\right)=\sum_{m=1}^{5}U_m\left(x,t\right)&=(1+t(1.0000028708038506+t(0.4999021407477205+t(0.16754988130644843\\ &+(0.03871980496628806+0.012126458064182619t)t))))x^2, \end{aligned}$$

which is approximate solution of the exact solution x^2e^t, and is match with the solution obtained by ADM [13], VIM [14] and HPM [5].

5 Conclusion

The results have been obtained by using the OHAM which are presented here agree well with the results obtained by ADM, VIM and HPM. The advantage of OHAM is that we can rapidly converges the series solution by taking optimal value of c_0, which is not possible in other analytic methods like ADM and VIM. Finally, we have concluded that the OHAM is very powerful and efficient analytical approximation method to solve such types of linear and nonlinear partial differential equations arise in many problems of fluid flow through porous media and oil recovery processes.

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Competing Interests

The authors declare that no competing interests exist.

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