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LQ Optimal Control-Based Variable Gain Controllers for Linear Systems with Structured Uncertainties and Its Performance Analysis

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Abstract

This paper proposes a variable gain controller for a linear system with structured uncertainties. The proposed variable gain controller is based on LQ optimal control for the nominal system and consists of the optimal feedback gain and a time-varying adjustable parameter which is designed so as to reduce the effect of uncertainties. The proposed LQ optimal control-based variable gain controller can achieve good transient performance which is close to LQ optimal control for the nominal system and adjust the magnitude of the control input. In this paper, we show sufficient conditions for the existence of the proposed variable gain robust controller for the uncertain linear system. Finally, an illustrative example is included.

Keywords: Variable gain robust controllers, LQ optimal control, compensation inputs, satisfactory transient behavior, control input adjustment.

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1 Introduction

In general, there exists a gap between mathematical models and practical systems, i.e. the design space and the real space. Therefore controller design methods dealing with the model uncertainties have been required and for dynamical systems with unknown parameters, a large number of design methods of robust state feedback controllers have been presented (e.g. [1, 2, 3] and references therein). For a system with structured uncertainties, several quadratic stabilizing control laws have also been suggested and a connection between quadratic stabilization and $\mathcal{H}^\infty$ control has been established[4]. It is well known that for robust control for linear dynamical systems with uncertainties, the concept of quadratic stabilization via fixed quadratic Lyapunov functions plays an important role in dealing with the controller design.

By the way in most practical situations, it is desirable to design robust control systems which achieve not only robust stability but also an adequate level of performance. Therefore many robust controllers achieving some robust performances such as guaranteed cost control, mixed $\mathcal{H}^\infty/\mathcal{H}^2$ control, robust $\mathcal{H}^2$ control and so on have been suggested (e.g.[5, 6, 7]). Note that these controllers have fixed structure and these design methods are worst case design. Additionally, synthesis problems of robust controllers with variable gain have also been tackled (e.g. [8, 9]). Yamamoto and Yamauchi[8] have proposed a design method of a robust controller with the ability to adjust control performance adaptively. In [9], a robust controller with adaptation mechanizm has been presented and the robust controller is tuned on-line based on the information about parameter uncertainties. Besides, Oya and Hagino have proposed robust controllers with adaptive compensation inputs[10, 11]. Although the robust controllers in [10, 11] can achieve not only asymptotical stability but also satisfactory transient behavior, these robust controllers include the additional dynamics of the nominal system. Namely, the structure of these robust controllers is more complex. In [12], we have suggested a variable gain robust controller based on LQ optimal control for a linear system with parameter structured uncertainties.

From these viewpoints, we propose a variable gain robust controller based on LQ optimal control for a class of uncertain linear systems. The proposed variable gain controller consists of optimal feedback gain designed by using the nominal system and an adjustable time-varying parameter. The adjustable parameter is designed so as to reduce the effect of uncertainties. The proposed variable gain controller can achieve good transient performance which is close to the desirable trajectory generated by the nominal closed-loop system. This paper is organized as follows. In Section 2, notation and useful lemmas which are used in this paper are shown, and in Section 3, we introduce the class of uncertain linear systems under consideration. Section 4 contains the main results. Sufficient conditions for the existence of the proposed variable gain robust controller are presented. Finally, numerical examples are included to illustrate the results developed in this paper.

2 Preliminaries

In this section, we show notations and useful and well-known lemmas which are used in this paper.

In the sequel, we use the following notation. For a matrix $\mathcal{A}$, the transpose and the inverse of the matrix $\mathcal{A}$ are denoted by $\mathcal{A}^T$ and $\mathcal{A}^{-1}$, respectively. Also, $H_e\{\mathcal{A}\}$ means $\mathcal{A}+\mathcal{A}^T$ and I_n represents n-dimensional identity matrix and the notation $\mathrm{diag}\,(\mathcal{A}_1,\cdots,\mathcal{A}_N)$ denotes a block diagonal matrix composed of matrices $\mathcal{A}_i$ for $i=1,\cdots,N$. For real symmetric matrices $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A}>\mathcal{B}$ $(\mathrm{resp.}\ \mathcal{A}\geq\mathcal{B})$ means that $\mathcal{A}-\mathcal{B}$ is positive (resp. nonnegative) definite matrix. For a vector $\alpha\in\mathbb{R}^n$, $||\alpha||$ denotes standard Euclidian norm and for a matrix $\mathcal{A}$, $||\mathcal{A}||$ represents its induced norm. For two sets $\mathcal{S}$ and $\mathcal{T}$, $\mathcal{S}\subset\mathcal{T}$ means that the set $\mathcal{S}$ is a subset of $\mathcal{T}$, and $\mathcal{S}^c$ is a complement of $\mathcal{S}$. The the symbols "$\stackrel{\triangle}{=}$" and "$\star$" denote equality by definition and symmetric blocks in matrix inequalities, respectively.

Furthermore, the following well-known lemmas are used in this paper.

Lemma 1. For arbitrary vectors λ and ξ and the matrices $\mathcal{G}$ and $\mathcal{H}$ which have appropriate dimensions, the following relation holds.

$$H_e\left\{\lambda^T\mathcal{G}\Delta(t)\mathcal{H}\xi\right\} \leq 2\left\|\mathcal{G}^T\lambda\right\|\left\|\mathcal{H}\xi\right\|$$

where $\Delta(t) \in \mathbb{R}^{p\times q}$ is a time-varying unknown matrix satisfying $\left\|\Delta(t)\right\| \leq 1.0$.

Proof. The above relation is easily obtained by Schwartz's inequality[13]. □

Lemma 2. ($\mathcal{S}$-procedure) Let $\mathcal{F}(x)$ and $\mathcal{G}(x)$ be two arbitrary quadratic forms over $\mathbb{R}^n$. Then $\mathcal{F}(x) < 0$ for $\forall x \in \mathbb{R}^n$ satisfying $\mathcal{G}(x) \leq 0$ if and only if there exist a nonnegative scalar τ such that

$$\mathcal{F}(x) - \tau\mathcal{G}(x) \leq 0 \ \text{ for } \ \forall x \in \mathbb{R}^n$$

Proof. See Boyd et al.[14] □

Lemma 3. (Schur complement) For a given constant real symmetric matrix $\varXi$, the following arguments are equivalent.

(i). $\varXi = \begin{pmatrix} \varXi_{11} & \varXi_{12} \\ \varXi_{12}^T & \varXi_{22} \end{pmatrix} > 0$

(ii). $\varXi_{11} > 0$ and $\varXi_{22} - \varXi_{12}^T\varXi_{11}^{-1}\varXi_{12} > 0$

(iii). $\varXi_{22} > 0$ and $\varXi_{11} - \varXi_{12}\varXi_{22}^{-1}\varXi_{12}^T > 0$

Proof. See Boyd et al.[14] □

3 Problem Formulation

Consider the uncertain linear system described by the following state equation (see **Remark 1**).

$$\frac{d}{dt}x(t) = (A + \mathcal{D}\Delta(t)\mathcal{E})\,x(t) + Bu(t) \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are the vectors of the state (assumed to be available for feedback) and the control input, respectively. In (3.1), the matrices A and B denote the nominal values of the uncertain system of (3.1). The matrices $\mathcal{D}$ and $\mathcal{E}$ which have appropriate dimensions represent the structure of uncertainties and the time-varying parameter $\Delta(t) \in \mathbb{R}^{p\times q}$ shows unknown parameters which satisfy $\left\|\Delta(t)\right\| \leq 1.0$. Beside the nominal system, ignoring the unknown parameters in (3.1), is given by

$$\frac{d}{dt}\overline{x}(t) = A\overline{x}(t) + B\overline{u}(t). \tag{3.2}$$

In this paper first of all, we consider the standard linear quadratic control problem for the nominal system of (3.2) in order to generate the desired response for the uncertain system of (3.1) systematically. Namely we define the following quadratic cost function for the nominal system of (3.2).

$$\mathcal{J} = \int_0^\infty \left(\overline{x}^T(t)\mathcal{Q}\overline{x}(t) + \overline{u}^T\mathcal{R}\overline{u}(t)\right)dt \tag{3.3}$$

where the matrices $\mathcal{Q} \in \mathbb{R}^{n\times n}$ and $\mathcal{R} \in \mathbb{R}^{m\times m}$ are positive definite. It is well-known that the optimal control input minimizing the quadratic cost function of (3.3) is given by $\overline{u}(t) = -K\overline{x}(t)$, where $K \in \mathbb{R}^{m\times n}$ represent the optimal control gain matrix. Note that the closed-loop system matrix $A_K \stackrel{\triangle}{=} A -$

BK is stable and the optimal feedback gain matrix $K \in \mathbb{R}^{m\times n}$ is derived as $K = \mathcal{R}^{-1}B^T\mathcal{P}$ where $\mathcal{P} \in \mathbb{R}^{n\times n}$ is unique solution of the algebraic Riccati equation

$$H_e\left\{A^T\mathcal{P}\right\} - \mathcal{P}B\mathcal{R}^{-1}B^T\mathcal{P} + \mathcal{Q} = 0. \tag{3.4}$$

Now by using the optimal feedback gain matrix $K \in \mathbb{R}^{m\times n}$ for the nominal system of (3.2), we consider the following control input.

$$u(t) \stackrel{\triangle}{=} \gamma(x,t)Kx(t) \tag{3.5}$$

where $\gamma(x,t) \in \mathbb{R}^1$ is a time-varying adjustable parameter so as to compensate the effect of unknown parameters. One can see from (3.1) and (3.5) that the following closed-loop system is obtained.

$$\frac{d}{dt}x(t) = Ax(t) + \mathcal{D}\Delta(t)\mathcal{E}x(t) + \gamma(x,t)BKx(t). \tag{3.6}$$

From the above discussion, our control objective in this paper is to design the robust stabilizing controller which achieves good transient performance for the uncertain closed-loop system of (3.6). That is to design the time-varying adjustable parameter $\gamma(x,t) \in \mathbb{R}^1$ such that the closed-loop system of (3.6) is robustly stable and achieves satisfactory transient performance close to LQ optimal control for the nominal system of (3.2).

Remark 1. In this paper, we consider the uncertain dynamical system of (3.1) which has uncertainties in the state matrix only. The proposed design scheme of the variable gain controller derived in next section can also be applied to the case that the uncertainties are included in both the system matrix and the input matrix. By introducing additional actuator dynamics and constituting an augmented system, uncertainties in the input matrix are embedded in the system matrix of the augmented system[15]. Therefore the same design procedure can be applied.

4 Main Results

In this section, we show a design method of the proposed variable gain controller such that the uncertain system of (3.1) is asymptotically stable.

The following theorem gives sufficient conditions for the existence of the proposed controller.

Theorem 1. Consider the uncertain linear system of (3.1) and the control input of (3.5).

For a given positive scalar δ which is a design parameter, if there exist the positive scalars τ_1 and τ_2 satisfying the LMI

$$\begin{pmatrix} -\mathcal{Q} - (1+\tau_1)\mathcal{P}B\mathcal{R}^{-1}B^T\mathcal{P} + \delta\tau_1 I_n + \tau_2\mathcal{E}^T\mathcal{E} & \mathcal{P}\mathcal{D} \\ \star & -\tau_2 I_p \end{pmatrix} < 0 \tag{4.1}$$

then the adjustable time-varying parameter $\gamma(t) \in \mathbb{R}^1$ is determined as

$$\gamma(x,t) = \begin{cases} -\left(1 + \dfrac{\left\|\mathcal{D}^T\mathcal{P}x(t)\right\|\left\|\mathcal{E}x(t)\right\|}{\left\|\mathcal{R}^{-1/2}B^T\mathcal{P}x(t)\right\|^2}\right) & \text{if} \quad x^T(t)\mathcal{P}B\mathcal{R}^{-1}B^T\mathcal{P}x(t) \geq \delta x^T(t)x(t) \\ -\left(1 + \dfrac{\left\|\mathcal{D}^T\mathcal{P}x(t)\right\|\left\|\mathcal{E}x(t)\right\|}{\delta x^T(t)x(t)}\right) & \text{if} \quad x^T(t)\mathcal{P}B\mathcal{R}^{-1}B^T\mathcal{P}x(t) \leq \delta x^T(t)x(t). \end{cases} \tag{4.2}$$

Then the uncertain closed-loop system of (3.6) is robustly stable.

Proof. By using the unique solution $\mathcal{P} \in \mathbb{R}^{n\times n}$ of the algebraic Riccati equation of (3.4), we consider the following quadratic function.

$$\mathcal{V}(x,t) \stackrel{\triangle}{=} x^T(t)\mathcal{P}x(t). \tag{4.3}$$

The time derivative of the quadratic function $\mathcal{V}(x,t)$ of (4.3) can be written as

$$\frac{d}{dt}\mathcal{V}(x,t) = x^T(t)\left[H_e\left\{A^T\mathcal{P}\right\}\right]x(t) + 2x^T(t)\mathcal{P}\mathcal{D}\Delta(t)\mathcal{E}x(t) + 2\gamma(t)x^T(t)\mathcal{P}BKx(t). \tag{4.4}$$

Since the matrix $\mathcal{P} \in \mathbb{R}^{n\times n}$ is the unique solution of the algebraic Riccati equation of (3.4), The time derivative of the quadratic function $\mathcal{V}(x,t)$ can be rewritten as

$$\frac{d}{dt}\mathcal{V}(x,t) = -x^T(t)\left(\mathcal{Q} - \mathcal{P}B\mathcal{R}^{-1}B^T\mathcal{P}\right)x(t) + 2x^T(t)\mathcal{P}\mathcal{D}\Delta(t)\mathcal{E}x(t) + 2\gamma(t)x^T(t)\mathcal{P}BKx(t). \tag{4.5}$$

Now, we consider the case of $x^T(t)\mathcal{P}B\mathcal{R}^{-1}B^T\mathcal{P}x(t) \geq \delta x^T(t)x(t)$. We see from **Lemma 1** and the relation $K = \mathcal{R}^{-1}B^T\mathcal{P}$ that the inequality for the time derivative of the quadratic function $\mathcal{V}(x,t)$ of (4.3)

$$\begin{aligned}\frac{d}{dt}\mathcal{V}(x,t) &\leq -x^T(t)\left(\mathcal{Q} - \mathcal{P}B\mathcal{R}^{-1}B^T\mathcal{P}\right)x(t) + 2\left\|\mathcal{D}^T\mathcal{P}x(t)\right\|\left\|\Delta(t)\mathcal{E}x(t)\right\| + 2\gamma(t)x^T(t)\mathcal{P}BKx(t)\\ &\leq -x^T(t)\left(\mathcal{Q} - \mathcal{P}B\mathcal{R}^{-1}B^T\mathcal{P}\right)x(t) + 2\left\|\mathcal{D}^T\mathcal{P}x(t)\right\|\left\|\mathcal{E}x(t)\right\|\\ &\quad + 2\gamma(t)x^T(t)\mathcal{P}B\mathcal{R}^{-1}B^T\mathcal{P}x(t).\end{aligned} \tag{4.6}$$

is satisfied. Besides, by using the adjustable time-varying parameter $\gamma(t)$ of (4.2), we find that the following relation holds.

$$\begin{aligned}\frac{d}{dt}\mathcal{V}(x,t) &\leq -\,x^T(t)\left(\mathcal{Q} + \mathcal{P}B\mathcal{R}^{-1}B^T\mathcal{P}\right)x(t)\\ &< 0 \quad \text{for} \quad \forall x(t) \neq 0.\end{aligned} \tag{4.7}$$

Next, we consider the case of $x^T(t)\mathcal{P}B\mathcal{R}^{-1}B^T\mathcal{P}x(t) < \delta x^T(t)x(t)$ and then the time derivative of the quadratic function $\mathcal{V}(x,t)$ of (4.5) can also be described as

$$\frac{d}{dt}\mathcal{V}(x,t) = \begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix}^T \left(\begin{array}{c|c} -\mathcal{Q} + \mathcal{P}B\mathcal{R}^{-1}B^T\mathcal{P} & \mathcal{P}\mathcal{D} \\ \hline \star & 0 \end{array}\right) \begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix} + 2\gamma(t)x^T(t)\mathcal{P}B\mathcal{R}^{-1}B^T\mathcal{P}x(t). \tag{4.8}$$

In (4.8), $\xi(t)$ is a $n \times \mathcal{N}$-dimentional vector given by

$$\xi(t) \stackrel{\triangle}{=} \Delta(t)\mathcal{E}x(t). \tag{4.9}$$

Note that from the relation $\left\|\Delta(t)\right\| \leq 1.0$ for the unknown parameter $\Delta(t) \in \mathbb{R}^{p\times q}$, the following inequality for the vector $\xi(t) \in \mathbb{R}^{n\times\mathcal{N}}$ is satisfied.

$$\xi^T(t)\xi(t) \leq x^T(t)\mathcal{E}^T\mathcal{E}x(t) \tag{4.10}$$

Therefore one can see that if the inequality condition

$$\begin{aligned}&\begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix}^T \left(\begin{array}{c|c} -\mathcal{Q} + \mathcal{P}B\mathcal{R}^{-1}B^T\mathcal{P} & \mathcal{P}\mathcal{D} \\ \hline \star & 0 \end{array}\right) \begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix} + 2\gamma(t)x^T(t)\mathcal{P}B\mathcal{R}^{-1}B^T\mathcal{P}x(t) < 0\\ &\text{s.t.} \quad x^T(t)\mathcal{P}B\mathcal{R}^{-1}B^T\mathcal{P}x(t) < \delta x^T(t)x(t) \text{ and } \xi^T(t)\xi(t) \leq x^T(t)\mathcal{E}^T\mathcal{E}x(t)\end{aligned} \tag{4.11}$$

holds, then the following inequality is also satisfied.

$$\frac{d}{dt}\mathcal{V}(x,t) < 0 \quad \text{for} \quad \forall x(t) \neq 0. \tag{4.12}$$

Thus we consider the condition of (4.11). Since the adjustable time-varying parameter $\gamma(t)$ of (4.2) takes nagative value and the matrix $\mathcal{P}B\mathcal{R}^{-1}B^T\mathcal{P}$ is positive definite, we have

$$\begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix} \left(\begin{array}{c|c} -\mathcal{Q} - \mathcal{P}B\mathcal{R}^{-1}B^T\mathcal{P} & \mathcal{P}\mathcal{D} \\ \hline \star & 0 \end{array} \right) \begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix} < 0$$
$$\text{s.t.} \quad x^T(t)\mathcal{P}B\mathcal{R}^{-1}B^T\mathcal{P}x(t) < \delta x^T(t)x(t) \text{ and } \xi^T(t)\xi(t) \le x^T(t)\mathcal{E}^T\mathcal{E}x(t). \tag{4.13}$$

The inequality of (4.13) is a sufficient condition for the inequality of (4.11). Namely, if the condition of (4.13) holds, then the inequality of (4.12) is also satisfied. Applying **Lemma 2** ($\mathcal{S}$-procedure) to the condition of (4.13) and some trivial manipulations give the LMI of (4.1). Therefore for the case of $x^T(t)\mathcal{P}B\mathcal{R}^{-1}B^T\mathcal{P}x(t) < \delta x^T(t)x(t)$, if the LMI of (4.1) is feasible then the relation of (4.12) is satisfied.

From the above discussion, the quadratic function $\mathcal{V}(x,t)$ becomes a Lyapunov function and the uncertain linear system of (3.1) is ensured to be stable. It follows that the result of the theorem is true. The proof of **Theorem 1** is completed. ☐

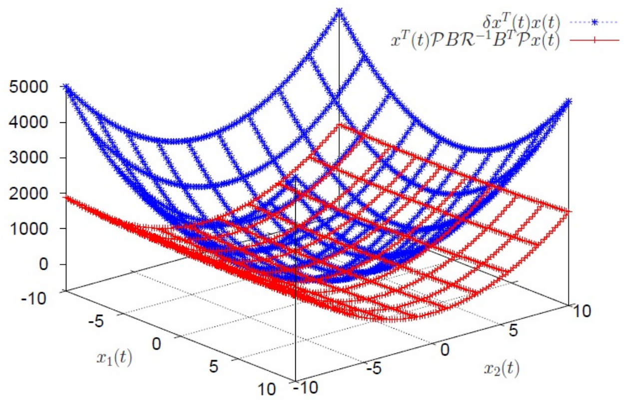

Figure 1: An example for the quadratic functions $x^T(t)\mathcal{P}B\mathcal{R}^{-1}B^T\mathcal{P}x(t)$ and $\delta x^T(t)x(t)$ (2-dimentional case)

Remark 2. The adjutable parameter $\gamma(x,t) \in \mathbb{R}^1$ in the proposed controller can be obtained as (4.2) provided that there exists the positive constants τ_1 and τ_2 in the LMI of (4.1). Note that these constants τ_1 and τ_2 does not utilized in the proposed controller. Additionally, in this paper, we introduce the design parameter $\delta > 0$ in (4.2) and this parameter δ plays important roles in the proposed control. One can easily find that for two positive constants δ^+ and δ^- which satisfy $\delta^+ > \delta^-$, the relation $\mathcal{X}_+ \subset \mathcal{X}_-$ holds where $\mathcal{X}_+$ and $\mathcal{X}_-$ denote subspaces defined as

$$\begin{aligned} \mathcal{X}_+ &\triangleq \left\{ x \in \mathbb{R}^n \mid x^T(t)\mathcal{P}B\mathcal{R}^{-1}B^T\mathcal{P}x(t) > \delta^+ x^T(t)x(t) \right\}, \\ \mathcal{X}_- &\triangleq \left\{ x \in \mathbb{R}^n \mid x^T(t)\mathcal{P}B\mathcal{R}^{-1}B^T\mathcal{P}x(t) > \delta^- x^T(t)x(t) \right\}. \end{aligned} \tag{4.14}$$

This fact means that if the parameter $\delta^- > 0$ is selected then the magnitude of the control input is large comparing with one for δ^+, and the decent for the quadratic function $\mathcal{V}(x,t)$ is close to $-x^T(t)\left(\mathcal{Q} + \mathcal{P}B\mathcal{R}^{-1}B^T\mathcal{P}\right)x(t)$ because $\mathcal{X}_+^c \supset \mathcal{X}_-^c$, i.e. the selection of the paramerer $\delta > 0$ can be regarded as determining the switching region in the state space (see Figure 1). Besides the quadratic function $\mathcal{V}(x, t)$ of (4.3) becomes a Lyapunov function for the uncertain system of (3.1) and the quadratic function $\mathcal{V}(\overline{x}, t)$ is also a Lyapnov function for the nominal closed-loop system. Note that the descent of the quadratic function $\mathcal{V}(\overline{x}, t)$ is described as $-\overline{x}^T(t)\left(\mathcal{Q} + \mathcal{P}B\mathcal{R}^{-1}B^T\mathcal{P}\right)\overline{x}(t)$. Thus in the case that $\delta^- > 0$ is chosen, good transient performance can be achieved and conversely, if the parameter $\delta^+ > 0$ is selected then the magnitude of the control input is suppressed, because the Lyapunov function for the uncertain system of (3.1) and one of the nominal system of (3.3) have same level set. Therefore, although there is a trade off between achieving good transient performance and avoiding excessive control input, by selecting the design parameter $\delta > 0$, the control performance can be adjusted, i.e. one can see that the proposed variable gain robust controller is useful.

Remark 3. In order to design the proposed control system, designers have to solve the LMI of (4.1). If the LMI of (4.1) is feasible for $\exists\delta > 0$ then one can easily see that the LMI of (4.1) is always satisfied for the positive scalar $\forall\delta^\star < \delta$. Namely, the transient performance and the magnitude of the control input can be improved by on-line adjustment for the design parameter δ. However, the on-line adjustment strategy for the design parameter δ has not been established and this problem is one of our future reseach subjects.

Remark 4. The proposed variable gain robust controller based on LQ optimal control for the nominal system has some advantages as follows. The proposed controller design approach is very simple, and by selecting the design parameter, the proposed variable gain robust controller can achieve good transient performance or the excessive control input can be avoided (see **Remark 1**, **Remark 2** and Section 5). Besides, the structure of the proposed control system is also simple comparing with the existing results for variable gain robust controllers (i.g. [10, 11]).

In [12], the linear system with parameter structured uncertainties is considered. If the number of the unknown parameters equals to $\mathcal{N}$, then the size of the LMI to be solved in [12] is $(\mathcal{N}+1)n \times (\mathcal{N}+1)n$. On the other hand in this paper, the size of LMIs of (4.1) equals to $2n \times 2n$. Namely, the proposed approach is rather suitable for the structured uncertainties than the parameter structured uncertainties.

5 Illustrative Examples

In order to demonstrate the efficiency of the proposed control scheme, we have run a simple example. The control problem considered here are not necessary practical. However, the simulation results stated below illustrate the distinct feature of the proposed variable gain controller.

Consider the uncertain linear system described as

$$\frac{d}{dt}x(t) = \begin{pmatrix} -2.0 & 1.0 \\ 0.0 & 1.0 \end{pmatrix} x(t) + \begin{pmatrix} 1.0 & 3.0 \\ 0.0 & 1.0 \end{pmatrix} \Delta(t) \begin{pmatrix} 0.0 & 0.0 \\ 0.0 & 2.0 \end{pmatrix} x(t) + \begin{pmatrix} 0.0 \\ 1.0 \end{pmatrix} u(t). \tag{5.1}$$

Firstly we select the weighting matrices $\mathcal{Q}$ and $\mathcal{R}$ such as $\mathcal{Q} = \mathrm{diag}(1.0, 9.0)$ and $\mathcal{R} = 1.0$ for the quadratic cost function for the standard linear quadratic control problem, respectively. Then solving the algebraic Riccati equation of (3.4), we obtain

$$\begin{aligned} K &= \begin{pmatrix} 4.81740 \times 10^{-2} & 4.17748 \end{pmatrix} \\ \mathcal{P} &= \begin{pmatrix} 2.49420 \times 10^{-1} & 4.81740 \times 10^{-2} \\ \star & 4.17748 \end{pmatrix}. \end{aligned} \tag{5.2}$$

In this example, we consider the following two kinds of the parameters for the design parameter $\delta \in \mathbb{R}^1$ in (4.2).

$$\begin{aligned} &\bullet\ \Sigma_1^* \ :\ \delta = 1.0 \times 10^2, \\ &\bullet\ \Sigma_2^* \ :\ \delta = 5.0 \times 10^4. \end{aligned} \tag{5.3}$$

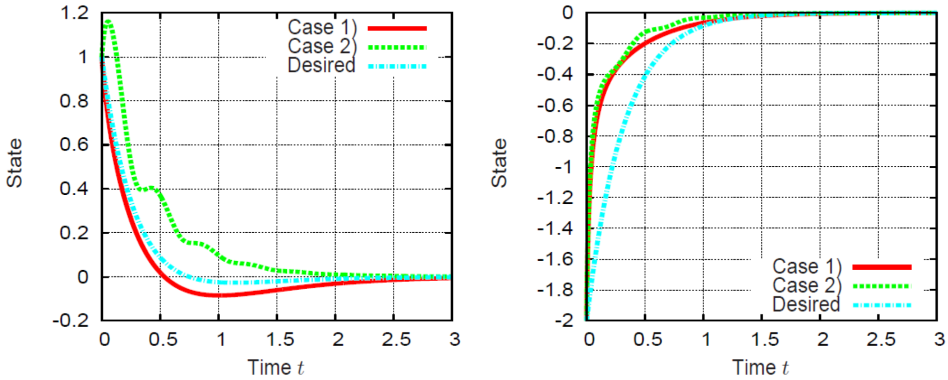

Figure 2: Time histories of the state $x_1(t)$: Σ_1^* (Left) and the state $x_2(t)$: Σ_1^* (Right)

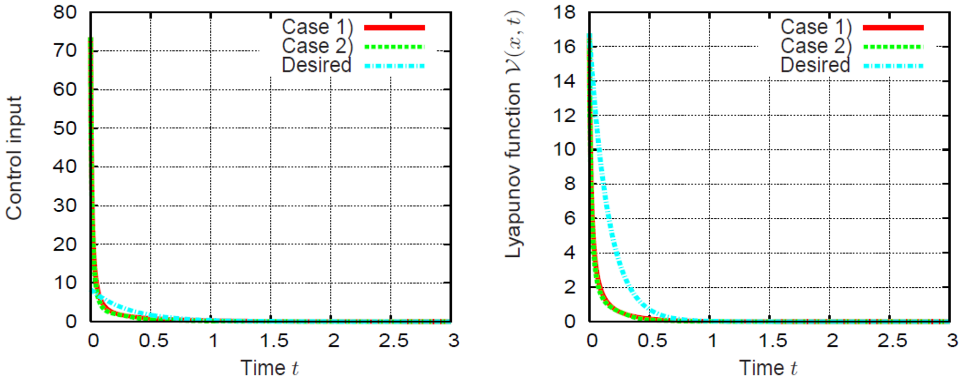

Figure 3: Time histories of the control input $u(t)$: Σ_1^* (Left) and time histories of the Lyapunov function $\mathcal{V}(x,t)$: Σ_1^* (Right)

For the these two design parameters, solving the LMI condition of (4.1), we have

$$\begin{aligned} &\bullet\ \Sigma_1^*\ : \tau_1 = 1.0 \times 10^{-7}, \quad \tau_2 = 1.51028, \\ &\bullet\ \Sigma_2^*\ : \tau_1 = 1.0 \times 10^{-7}, \quad \tau_2 = 1.51418. \end{aligned} \tag{5.4}$$

Now in this example, we consider the following two cases for the unknown parameters.

$$\begin{aligned} &\bullet\ \text{Case 1)} \quad : \quad \Delta(t) = \begin{pmatrix} 4.67360 & -5.96857 \\ 1.41379 & 4.81654 \end{pmatrix} \times 10^{-1} \\ &\bullet\ \text{Case 2)} \quad : \quad \Delta(t) = \text{diag}\left(\sin(5\pi t)\ ,\ -\cos(5\pi t) \right) \end{aligned} \tag{5.24}$$

Furthermore, the initial values for the uncertain system of (5.1) and its nominal system are selected as $x(0) = \overline{x}(0) = \begin{pmatrix} 1.0 & -2.0 \end{pmatrix}^T$.

The results of the simulation of this example are depicted in Figures 2 - 5. In these figures, "Case 1)" and "Case 2)" represent the time-histories of the state variables $x_1(t)$ and $x_2(t)$, the control input $u(t)$ and the Lyapunov function $\mathcal{V}(x,t)$. Besides, "Desired" represents the desirable transient behavior, the control input and the time-histories of the Lyapunov function $\mathcal{V}(x,t)$ generated by the

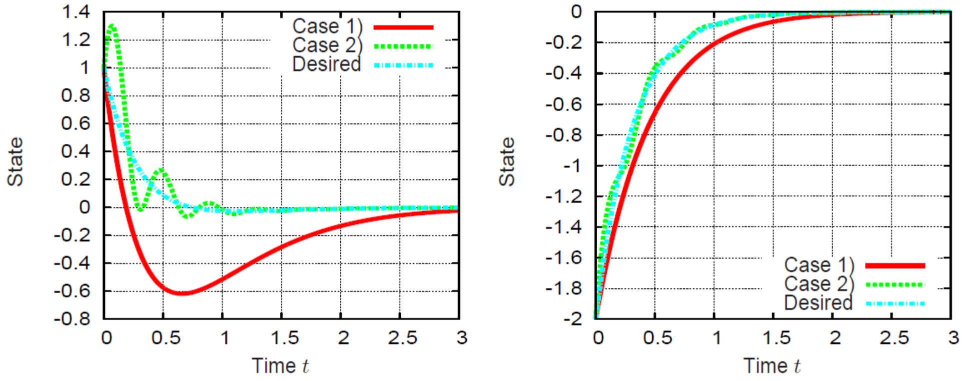

Figure 4: Time histories of the state $x_1(t)$: Σ_2^* (Left) and the state $x_2(t)$: Σ_2^* (Right)

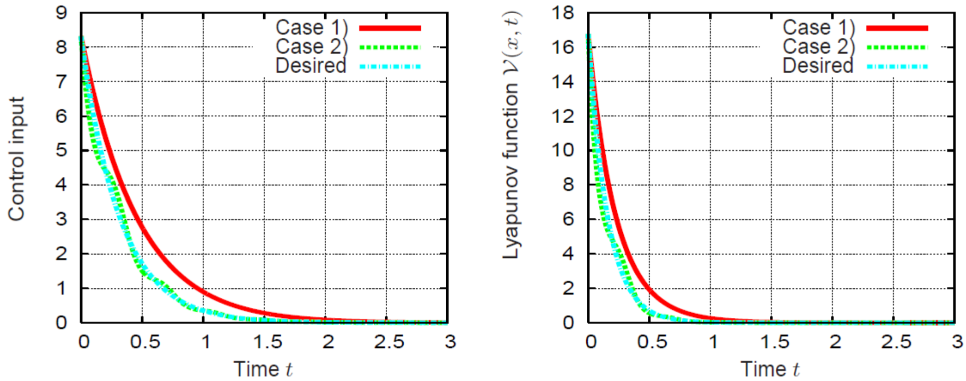

Figure 5: Time histories of the control input $u(t)$: Σ_2^* (Left) and the Lyapunov function $\mathcal{V}(x,t)$: Σ_2^* (Right)

nominal system. Besides, in order to compare the proposed LQ optimal control-based variable gain controller with the LQ optimal control for the nominal system, we consider the following performance indeces $\mathcal{J}_x$ and $\mathcal{J}_u$.

$$\begin{aligned} \mathcal{J}_x &\triangleq \int_0^\infty x^T(t)\mathcal{Q}x(t)dt, \\ \mathcal{J}_u &\triangleq \int_0^\infty u^T(t)\mathcal{R}u(t)dt. \end{aligned} \tag{5.25}$$

The optimal value $\mathcal{J}_{\rm opt}$ of the quadratic cost function $\mathcal{J}$ of (3.3) can be computed as $\mathcal{J}_{\rm opt} = 16.76662$ and the performance indeces corresponding to $\mathcal{J}_x$ and $\mathcal{J}_u$ are given by

$$\begin{aligned} \mathcal{J}_{\overline{x}} &\triangleq \int_0^\infty \overline{x}^T(t)\mathcal{Q}\overline{x}(t)dt = 5.81942, \\ \mathcal{J}_{\overline{u}} &\triangleq \int_0^\infty \overline{u}^T(t)\mathcal{R}\overline{u}(t)dt = 1.09472 \times 10^1. \end{aligned} \tag{5.26}$$

Table 1: The performance indeces $\mathcal{J}_x$ and $\mathcal{J}_u$ and the optimal value for the nominal system : Σ_1^*

	$\mathcal{J}_x$	$\mathcal{J}_u$
Case 1)	1.66825	4.57275$\times 10^1$
Case 2)	1.59153	4.06920$\times 10^1$

Table 2: The performance indeces $\mathcal{J}_x$ and $\mathcal{J}_u$ and the optimal value for the nominal system : Σ_2^*

	$\mathcal{J}_x$	$\mathcal{J}_u$
Case 1)	8.48345	1.57691$\times 10^1$
Case 2)	5.55297	1.09498$\times 10^1$

From Figures 2 and 3, we see that the proposed variable gain controller (Σ_1^*) for Case 1) achieves good transient performance comparing with the controller (Σ_2^*) for Case 1). However, the proposed control input is excessive comparing with the nominal system. Additionally, for the proposed controller (Σ_1^*) one can see from Table 1 that the performance index $\mathcal{J}_x$ (resp. $\mathcal{J}_u$) is smaller (resp. larger) than the optimal value for the nominal system. On the other hand, one can see from Figures 4 and 5 that although the error between the transient response for the proposed variable gain controller (Σ_2^*) and the one of the nominal system is large for Case 1), the control input in Σ_2^* is close to the desired one. For Case 2) the proposed variable gain controller (Σ_2^*) achieves good transient response and the satisfactory control input as closely as possible to the optimal trajectory generated by the nominal system. We see from Table 2 that the performance indeces for Case 2) for the proposed robust controller (Σ_2^*) are good values. Namely, the proposed variable gain controller can adjust the transient performance and the control input by means of selecting the design parameter $\delta \in \mathbb{R}^1$ in (4.2). Therefore the effectiveness of the proposed variable gain controller is shown.

6 Conclusions

In this paper on the basis of our previous work[12] we have proposed an LQ optimal control-based variable gain robust controller for a linear system with structured uncertainties. Besides, by numerical simulations, the effectiveness of the proposed controller has been presented.

The proposed LQ optimal control-based variable gain controller has some advantages. One can see that the crucial difference between the existing results[10, 11] and our new one is that the structure of proposed controller is simple and the proposed variable gain controller can adjust the transient performance. Namely although there is a trade off between achieving good transient performance and avoiding excessive control input, by selecting the design parameter $\delta > 0$, the control performance can be adjusted. Additionally, we have discussed the performance for the proposed variable gain robust controller.

The future research subjects are an extension of the proposed controller to such a broad class of systems as uncertain large-scale systems, uncertain discrete-time systems, uncertain time-delay systems and so on. In addition, the on-line adjustment strategy of the design parameter δ for achieving good transient performance is also our future research subject.

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Competing Interests

The authors declare that no competing interests exist.

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