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Feedback Controls for Finite Time or Asymptotic Compensation in Lumped Disturbed Systems

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Abstract

In this paper, we study the possibility of finite time or asymptotic compensation of disturbances f, for a class of linear lumped systems

$$\begin{cases} \dot{z}(t) &= Az(t) + Bu(t) + f(t) \\ z(0) &= z_0 \end{cases}$$

augmented with the output equation

$$y(t) = Cz(t)$$

using directly feedback controls $u = K\ f$, where K is a linear operator. We give appropriate definitions and characterization results of this notion, called K-remediability. We also examine the relationship with the remediability as studied in previous works. Illustrative examples are presented and various situations are considered. The relation $u = K\ f$ includes the usual form $\overline{u} = \overline{K}\ y_f$, where y_f is the term corresponding, in the expression of the observation y, to the disturbance f. In the linear case, y_f may be deduced easily from the observation y, even if f is not known.

Keywords: Lumped disturbed systems, remediability, feedback control, observation.
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1 Introduction and Problem Statement

1.1 Introduction

Let us consider without loss of generality, a class of finite dimension linear dynamical systems described by the following state equation

$$(S)\begin{cases} \dot{z}(t) &= Az(t)+Bu(t)+f(t) \;;\; 0<t<T \\ z(0) &= z_0 \end{cases} \tag{1.1}$$

augmented with the following output equation

$$(E)\;\; y(t)=Cz(t) \tag{1.2}$$

with $A\in M_n(\mathbb{R})\equiv M_{n,n}(\mathbb{R})$, $B\in M_{n,p}(\mathbb{R})$, $C\in M_{q,n}(\mathbb{R})$; $n,\ p,\ q\geq 1$. $f\in L^2(0,T;\mathbb{R}^n)$; $u\in L^2(0,T;\mathbb{R}^n)$. The observation at the final time T is given by

$$\begin{aligned} y(T) &= Ce^{AT}z_0+CH_Tu+C\overline{H}_Tf \\ &= Ce^{AT}z_0+CH_Tu+y_f(T) \end{aligned}$$

where

$$\begin{aligned} H_T \;:\; L^2(0,T;\mathbb{R}^p) &\rightarrow \mathbb{R}^n \\ u &\rightarrow H_Tu=\textstyle\int_0^T e^{A(T-t)}Bu(t)dt \end{aligned} \tag{1.3}$$

$$\begin{aligned} \overline{H}_T \;:\; L^2(0,T;\mathbb{R}^n) &\rightarrow \mathbb{R}^n \\ f &\rightarrow \overline{H}_Tf=\textstyle\int_0^T e^{A(T-t)}f(t)dt \end{aligned} \tag{1.4}$$

and

$$y_f(t)=\int_0^t Ce^{A(t-s)}f(s)ds \tag{1.5}$$

In the case where the system is autonomous ($f=0$ and $u=0$), the observation is normal. At the final time, it is given by $y(T)=Ce^{AT}z_0$. But if $f\neq 0$ and $u=0$, generally

$$y(T)\neq Ce^{AT}z_0$$

The problem of compensation (remediability) is to study the existence of a control operator B ensuring the compensation of the effect of any disturbance f.

Hence, the system (S), augmented with the output (E) (or $(S)+(E)$, or also indifferently (A,B,C)) is remediable on $[0,T]$, if for any $f\in L^2(0,T;\mathbb{R}^n)$, there exists $u\in L^2(0,T;\mathbb{R}^p)$ such that

$$\int_0^T Ce^{A(T-s)}Bu(s)ds+\int_0^T Ce^{A(T-s)}f(s)ds=0$$

i.e.

$$CH_Tu+R_C(T)f=0$$

where

$$R_C(t)f=C\overline{H}_tf=y_f(t)$$

The problem of remediability as defined above for finite (as well as for infinite) dimension linear systems, has been studied. Characterization results have been established for various types of systems [Afifi, L. et al. ([1], [2], [3], [4], [5], [6])], applications were given and different situations have been considered. In each case, and under convenient hypothesis, it has been shown how to find, from the observation only, a control u_f ensuring the exact compensation of a disturbance f. Such a control depends certainly on f, but the nature of this dependance is not generally obvious.

1.2 Problem statement

In this work, we consider a natural question which consists to examine this relation. More precisely, we study the possibility of compensation using controls

$$u = K\ f \tag{1.6}$$

depending linearly on the disturbance, that is to say, as a feedback of the disturbance (or its corresponding observation y_f). Hence, the problem consists to study, with respect to the matrices A, B and C, the existence of an operator

$$K : L^2(0, T; \mathbb{R}^n) \to L^2(0, T; \mathbb{R}^p)$$

such that for any disturbance $f \in L^2(0, T; \mathbb{R}^n)$, we have

$$\int_0^T Ce^{A(T-s)} f(s)ds + \int_0^T Ce^{A(T-s)} BKf(s)ds = 0$$

which is equivalent to

$$C\overline{H}_T\ f + C\overline{H}_T BKf = 0$$

or

$$C\overline{H}_T\ [I + BK]f = 0$$

Let us note that a disturbance f may be known or unknown. However in the linear case, its corresponding observation y_f given by

$$y_f(t) = y(t) - Ce^{At}z_0 - CH_t u$$

is generally known. Hence, the relation (1.6) is not restrictive and includes practical cases where the controls are of the form $\overline{u} = \overline{K}y_f$ and $\tilde{u} = \tilde{K}R_C(T)f$. We define and we characterize this notion called K-remediability. We also study its relationship with the classical remediability studied in previous works. An extension to the asymptotic case is also presented. In both cases, properties are given and various situations are considered.

The developments and results presented in this paper can be extended to linear distributed systems [Afifi, L. et al [6], Balakrishnan, A.V. [7], Curtain, R.F. and Pritchard, A.J. [8], Curtain, R.F. and Zwart, H.J. [9], El Jai, A. [10], Lions, J.-L. [11]] and also to other situations (non linear systems, delayed systems, ...).

2 K-remediability

We have the following definitions.

Definition 2.1.
i) A disturbance $f \in L^2(0, T; \mathbb{R}^n)$ is said to be K-remediable if

$$C\overline{H}_T(I + BK)f = 0$$

i.e.

$$f \in ker[C\overline{H}_T(I + BK)]$$

The operator K do not depend on the disturbance f.

ii) If F is a subspace of $\mathcal{E} \equiv L^2(0,T;\mathbb{R}^n)$, we say that the system $(S)+(E)$ is K-remediable on F, if any disturbance $f \in F$ is K-remediable, this is equivalent to

$$F \subset ker[C\overline{H}_T(I+BK)]$$

iii) If $F=\mathcal{E}$, we say that the system $(S)+(E)$ is K- remediable on $\mathcal{E}$, or simply K-remediable. In this case, we have

$$ker[C\overline{H}_T(I+BK)] = \mathcal{E}$$

or

$$C\overline{H}_T(I+BK) = 0 \text{ on } \mathcal{E}$$

The notion of K-remediability on the whole space $\mathcal{E}$ as defined above in part iii), may seems strong. But as it will be seen in the next section, it is not stronger than the remediability in the open loop case, which itself is weaker than the notion of controllability.

Moreover, the restriction of the problem to a subspace (or a subset) F of $\mathcal{E}$ is more flexible and allows to examine the problem of remediability for particular disturbances (constant, piecewise constant, periodic, sinusoidal, of the form $f(t) = Dv(t)$, ...). One can also consider the case of disturbances $f \in L^2(0,T;G)$, where G is a subspace (or a subset) of $\mathbb{R}^n$, and consequently the case where the disturbance involves only some variables (components) among n.

We have the following properties.

Proposition 2.1.
1) If the system $(S)+(E)$ is K-remediable on F_1, then it is on every $F_2 \subset F_1$.
2) Particularly, if $(S)+(E)$ is K-remediable, then it is K-remediable on any subspace F of $\mathcal{E}$.
3) In each case, the converse is not generally true.

Proof.
- The properties 1) and 2) derive from the definition.
- The property 3) is illustrated in example 3.1 □

Let us note that for controls of type $u = \widetilde{K}R_C(T)f$, definitions, properties and results are analogous by replacing the operator K by $KR_C(T)$. In this case, if $f \in ker(I+BKR_C(T)$, then

$$C\overline{H}_T[I+BKR_C(T)]f = 0$$

The converse is not necessarily true. This is illustrated in the following example.

Example 2.1. *We consider the case where*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \;\; ; \;\; I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ *with* $f_2 \neq 0$*;* $C=(1,0)$*;* $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *and* $K=1$.

It is easy to show that

$$[I+BKR_C(T)]\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

One can assume without loss of generality, that F is a closed subspace (or just a subset) of $\mathcal{E}$.

However

$$C\overline{H}_T[I + BKR_C(T)]\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

In the next section, we examine the problem of existence of an operator K ensuring the K-remediability of (S) + (E), that is to say, on the whole space $\mathcal{E}$, and hence its relation with the "open-loop" remediability.

3 Remediability and K-remediability

Under the remediability assumption, and using Hilbert Uniqueness Method, we show that such an operator K exists. The converse will be then examined. Let us first give the following characterization of the remediability which depends on the matrices A, B and C [Afifi, L. et al [5]].

Proposition 3.1. *$(S) + (E)$ is remediable on $[0, T]$ if and only if*

$$rank(CB \quad CAB \quad ... \quad CA^{n-1}B) = rank(C) \tag{3.1}$$

Proof. Since $Im(CH_T) \subset Im(R_C(T))$, then $(S)+(E)$ is remediable on $[0, T]$ if and only if $Im(R_C(T)) \subset Im(CH_T)$, or equivalently

$$ker((H_T)^* C^{tr}) \subset ker((R_C(T))^*)$$

Hence, using Caylay-Hamilton theorem and the fact that $(H_T)^* = B^{tr}e^{A^{tr}(T-.)}$, we have

$$\theta \in ker((H_T)^* C^{tr}) \iff \begin{pmatrix} B^{tr}C^{tr} \\ B^{tr}A^{tr}C^{tr} \\ \vdots \\ B^{tr}(A^{tr})^{n-1}C^{tr} \end{pmatrix} \theta = 0$$

On an other hand, we have $ker((R_C(T))^*) = ker(C^{tr})$, consequently the remediability of $(S) + (E)$ is equivalent to

$$ker \begin{pmatrix} B^{tr}C^{tr} \\ B^{tr}A^{tr}C^{tr} \\ \vdots \\ B^{tr}(A^{tr})^{n-1}C^{tr} \end{pmatrix} = ker(C^{tr})$$

we then have the result.

□

Concerning the existence of K, we have the following result.

Proposition 3.2. *If the system (S), augmented with the output (E), is remediable on $[0, T]$, then it is K-remediable on $[0, T]$, with*

$$K = -(H_T)^* C^{tr} (\Lambda_T)^{-1} R_C(T)$$

where Λ_T is the isomorphism defined from $\mathbb{R}^q$ to $\mathbb{R}^q$ by

$$\Lambda_T \, \theta = \int_0^T Ce^{A(T-s)}BB^{tr}e^{A^{tr}(T-s)}C^{tr}\theta ds = CH_T(H_T)^\star \, C^{tr}\theta$$

and in general, M^{tr} is the transposal matrix of M.

Proof. For $\theta \in \mathbb{R}^n$, we define on $\mathbb{R}^q$ the semi-norm

$$\|\theta\|_{\mathbb{R}^q} = \left(\int_0^T \left\|B^{tr} e^{A^{tr}(T-s)} C^{tr}\theta\right\|^2_{\mathbb{R}^n} ds\right)^{\frac{1}{2}}$$

We assume that $\|.\|_{\mathbb{R}^q}$ is a norm, or equivalently $ker((H_T)^* C^{tr}) = \{0\}$, using the fact that $B^{tr} e^{A^{tr}(T-.)} C^{tr} = (H_T)^* C^{tr}$. (A, B, C) is then remediable on $[0, T]$.

Under this condition, $\mathbb{R}^q$ is a Hilbert space with the inner product

$$\langle \theta, \tau \rangle_{\mathbb{R}^q} = \int_0^T \left\langle B^{tr} e^{A^{tr}(T-s)} C^{tr}\theta, B^{tr} e^{A^{tr}(T-s)} C^{tr}\tau \right\rangle ds; \forall \theta, \tau \in \mathbb{R}^q$$

and the operator $\Lambda_T \ : \ \mathbb{R}^q \ \longrightarrow \ \mathbb{R}^q$ is an isomorphism such that :

$$\langle \Lambda_T \theta, \tau \rangle_{\mathbb{R}^q} = \langle \theta, \tau \rangle_{\mathbb{R}^q} \ \ ; \ \ \forall \theta, \tau \in \mathbb{R}^q \text{ and } \|\Lambda_T \theta\|_{\mathbb{R}^q} = \|\theta\|_{\mathbb{R}^q} ; \forall \theta \in \mathbb{R}^q$$

Moreover, if $f \in L^2(0, T; \mathbb{R}^n)$, then there exists a unique θ_f in $\mathbb{R}^q$ such that

$$\Lambda_T \theta_f = -R_C(T) f$$

and the control u_{θ_f} defined by

$$u_{\theta_f}(s) = B^{tr} e^{A^{tr}(T-s)} C^{tr} \theta_f(s) = H^* C^{tr} \theta_f \ ; \ 0 < s < T$$

satisfies

$$R_C(T) f + C H_T u_{\theta_f} = 0$$

and is optimal with $\left\|u_{\theta_f}\right\|_{L^2(0,T;\mathbb{R}^p)} = \|\theta_f\|_{\mathbb{R}^q}$.
Since $\theta_f = -(\Lambda_T)^{-1} R_C(T) f$, we have

$$u_{\theta_f} = -(H_T)^* C^{tr} (\Lambda_T)^{-1} R_C(T) f$$

Consequently, the operator

$$K = \ - (H_T)^* C^{tr} (\Lambda_T)^{-1} R_C(T)$$

ensures the compensation of any disturbance $f \in L^2(0, T; \mathbb{R}^n)$, i.e., the K-remediability of (S) + (E). □

Concerning the converse, we have the following result.

Proposition 3.3. *If the system (S), augmented with the output (E), is K-remediable on $[0, T]$, then it is remediable on $[0, T]$.*

Proof. If (S) + (E) is K-remediable on $[0, T]$, then

$$ker[C\overline{H}_T(I + BK)] = \mathcal{E}$$

and hence, for any $f \in L^2(0, T; \mathbb{R}^n)$, we have

$$C\overline{H}_T f \ + C\overline{H}_T BKf = 0$$

or equivalently $C\overline{H}_T f \ + CH_T Kf = 0$. Therefore

$$C\overline{H}_T f \ + CH_T u = 0$$

where $u = Kf \in L^2(0, T; \mathbb{R}^p)$. (S) + (E) is then remediable on $[0, T]$. □

Let us note the following.

Remark 3.1. The system (S), augmented with the output equation (E) may be K-remediable on a subspace F, without being remediable. This situation is illustrated in the following example.

Example 3.1. *We consider the case where* A, B *and* C *are such that*

$$rank(CB \;\; CAB \;\; .... \;\; CA^{n-1}B) \neq \; rank(C)$$

for example that where $n = 2$, $p = 1$, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The system $(S) + (E)$ *is not remediable according to proposition 3.1. Consequently, using proposition 3.3, it is not* K*-remediable for every operator*

$$K : L^2(0, T; \mathbb{R}^n) \to L^2(0, T; \mathbb{R}^p)$$

However, for a given operator K*, it is* K*-remediable on the subspace*

$$F = ker[C\overline{H}_T(I + BK)]$$

Let us note also that in the case of disturbances of the form

$$f(t) = Dv(t)$$

with $D \in M_{n,r}(\mathbb{R})$ and $v \in L^2(0, T; \mathbb{R}^r)$, the definitions and the results are similar by replacing the operator $\overline{H}_T$ by $\overline{H}_T D$ and $C\overline{H}_T(I + BK)$ by $C\overline{H}_T(I + BK)D$.

We give hereafter, an extension to the case of an infinite time horizon.

4 Asymptotic Case

In this section, we give an asymptotic analysis of the problem. We consider the system:

$$(S_\infty) \begin{cases} \dot{z}(t) & = & Az(t) + Bu(t) + f(t) \; ; \; t > 0 \\ z(0) & = & z_0 \end{cases}$$

augmented with the output equation :

$$(E_\infty) \;\; y(t) = Cz(t) \; ; \; t > 0$$

with $A \in M_n(\mathbb{R})$, $B \in M_{n,p}(\mathbb{R})$, $C \in M_{q,n}(\mathbb{R})$; $f \in L^2(0, +\infty; \mathbb{R}^n)$, $u \in L^2(0, +\infty; \mathbb{R}^p)$. We have $y \in L^2(0, +\infty; \mathbb{R}^q)$.

Remark 4.1. As seen in the previous section, the finite time compensation problem is equivalent to:

For any $f \in L^2(0, T; \mathbb{R}^n)$, does exists a control $u \in L^2(0, T; \mathbb{R}^p)$ such that

$$\int_0^T Ce^{A(T-s)}Bu(s)ds + \int_0^T e^{A(T-s)}f(s)ds = 0$$

or equivalently

$$\int_0^T Ce^{At}Bv(t)ds + \int_0^T e^{At}g(t)dt = 0 \tag{4.1}$$

where $g(t) = f(T-t)$ and $v(t) = u(T-t)$. Consequently, the finite time remediability of $(S)+(E)$ can be also formulated as follows:

For any $g \in L^2(0,T;\mathbb{R}^n)$, does exists a control $v \in L^2(0,T;\mathbb{R}^p)$ satisfying (4.1) ?

Concerning the problem of asymptotic compensation [Afifi, L. et al ([4], [6])], let us recall that the system (S_∞), augmented with (E_∞), is remediable asymptotically if for any disturbance $f \in L^2(0,+\infty;\mathbb{R}^n)$, there exists a control $u \in L^2(0,+\infty;\mathbb{R}^p)$ such that

$$C\overline{H}_\infty f + CH_\infty u = 0$$

where

$$H_\infty u = \int_0^{+\infty} e^{At}Bu(t)dt$$

and

$$\overline{H}_\infty f = \int_0^{+\infty} e^{At}f(t)dt$$

Note that the operators $\overline{H}_\infty$ and H_∞ are not generally well defined. They are, if and only if, the system is exponentially stable [Afifi, L. et al ([5], [6]), Balakrishnan, A.V. [7], Curtain, R.F. and Pritchard, A.J. [8], Curtain, R.F. and Zwart, H.J. [9]], i.e.

$$\sup_{i=1,n} Re(\lambda_i) < 0$$

where λ_1, λ_2, ..., λ_n are the eigenvalues of A. This condition is not necessary for the study of the asymptotic compensation because the considered problem depends also on the choice of the output operator C [Afifi, L. et al ([4], [5], [6])]. In fact, we are concerned with the operators H_C^∞ and R_C^∞ defined by

$$H_C^\infty u = \int_0^{+\infty} Ce^{At}Bu(t)dt$$

and

$$R_C^\infty f = \int_0^{+\infty} Ce^{At}f(t)dt$$

which may be well defined even if $\overline{H}_\infty$ and H_∞ are not. This is illustrated in the following example.

Example 4.1. *We consider a two dimension system with*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \;\; ; \;\; B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \;\; and \;\; C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in L^2(0,+\infty;\mathbb{R}^2)$ *and* $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in L^2(0,+\infty;\mathbb{R}^2)$.

We have

$$\begin{aligned} H_C^\infty u &= \int_0^{+\infty} Ce^{At}Bu(t)dt = \int_0^{+\infty} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} u_1(t) + 2u_2(t) \\ 3u_1(t) + 4u_2(t) \end{bmatrix} dt \\ &= \int_0^{+\infty} \begin{bmatrix} 0 & e^{-t} \end{bmatrix} \begin{bmatrix} u_1(t) + 2u_2(t) \\ 3u_1(t) + 4u_2(t) \end{bmatrix} dt = \int_0^{+\infty} (3u_1(t) + 4u_2(t))e^{-t}dt \end{aligned}$$

which is finite since u_1 *and* $u_2 \in L^2(0,+\infty;\mathbb{R})$.

For $f \in L^2(0, +\infty; \mathbb{R}^2)$, we have

$$R_C^\infty f = \int_0^{+\infty} Ce^{At} f(t)dt = \int_0^{+\infty} \begin{bmatrix} 0 & e^{-t} \end{bmatrix} \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} dt = \int_0^{+\infty} f_2(t)e^{-t}dt$$

which is convergent (finite). Consequently, the operators H_C^∞ and R_C^∞ depending on C, are well defined even if the system is not exponentially stable.

The system (S_∞) + (E_∞), is asymptotically remediable if for any $f \in L^2(0, +\infty; \mathbb{R}^n)$, there exists $u \in L^2(0, +\infty; \mathbb{R}^p)$ such that

$$R_C^\infty f + H_C^\infty u = 0$$

This is equivalent, as in the case of a finite time horizon, to

$$rank(CB \;\; CAB \;\; .... \;\; CA^{n-1}B) = rank(C)$$

The proof is similar to that of proposition 4.1. Here also, we study with respect to the matrices A, B and C, the existence of an operator

$$K : L^2(0, +\infty; \mathbb{R}^n) \to L^2(0, +\infty; \mathbb{R}^p)$$

such that for any $f \in L^2(0, +\infty; \mathbb{R}^n)$, the control $u = K\ f$ ensures asymptotically, the compensation of the disturbance f. We give hereafter the corresponding definitions.

Definition 4.1.

i) A disturbance $f \in L^2(0, +\infty; \mathbb{R}^n)$ is said to be K-remediable asymptotically if

$$f \in ker[R_C^\infty + H_C^\infty K]$$

ii) If F is a subspace (or a subset) of $\mathcal{E}_\infty \equiv L^2(0, +\infty; \mathbb{R}^n)$, we say that the system $(S_\infty) + (E_\infty)$ is K-remediable asymptotically on F, if any disturbance $f \in F$ is K-remediable asymptotically, or equivalently

$$F \subset ker[R_C^\infty + H_C^\infty K]$$

iii) If $F = \mathcal{E}_\infty$, we say that $(S_\infty) + (E_\infty)$ is asymptotically K-remediable on $\mathcal{E}_\infty$, or simply K-remediable asymptotically. Then we have

$$ker[R_C^\infty + H_C^\infty K] = \mathcal{E}_\infty$$

This is equivalent to $R_C^\infty + H_C^\infty K \;=\; 0$ on $\mathcal{E}_\infty$.

In the last case where $F = \mathcal{E}_\infty$, the following result shows the relation between the global asymptotic remediability and the existence of an operator K ensuring the K-remediability.

Proposition 4.1.

1) If $(S_\infty) + (E_\infty)$, is asymptotically remediable, then it is K-remediable asymptotically with

$$K = -\,(H_C^\infty)^*(\Lambda_\infty)^{-1} R_C^\infty$$

where Λ_∞ is the bijection defined from $\mathbb{R}^q$ to $\mathbb{R}^q$ by

$$\Lambda_\infty \theta = \int_0^{+\infty} Ce^{At}BB^{tr}e^{A^{tr}t}C^{tr}\theta dt \;=\; H_C^\infty (H_C^\infty)^* \theta$$

2) Conversely, if there exists an operator $K : L^2(0, +\infty; \mathbb{R}^n) \to L^2(0, +\infty; \mathbb{R}^p)$ such that $(S_\infty) + (E_\infty)$ is asymptotically K-remediable, then $(S_\infty) + (E_\infty)$ is asymptotically remediable.

Proof. :

1) For $\theta \in \mathbb{R}^n$, we consider the semi-norm

$$\|\theta\|_{\mathbb{R}^q} = (\int_0^{+\infty} \left\| B^{tr} e^{A^{tr} t} C^{tr} \theta \right\|_{\mathbb{R}^n}^2 ds)^{\frac{1}{2}}$$

$\|.\|_{\mathbb{R}^q}$ is a norm if and only if, $(S_\infty) + (E_\infty)$ is asymptotically remediable. We suppose that $\|.\|_{\mathbb{R}^q}$ is a norm, then $\mathbb{R}^q$ is a Hilbert space with the inner product

$$\langle \theta, \tau \rangle_{\mathbb{R}^q} = \int_0^{+\infty} \left\langle B^{tr} e^{A^{tr} t} \mathsf{C}^{tr} \theta, B^{tr} e^{A^{tr} t} C^{tr} \tau \right\rangle ds; \forall \theta, \tau \in \mathbb{R}^q$$

and the operator Λ_∞ is an isomorphism from $\mathbb{R}^q$ to $\mathbb{R}^q$ such that :

$$\langle \Lambda_\infty \theta, \tau \rangle_{\mathbb{R}^q} = \langle \theta, \tau \rangle_{\mathbb{R}^q} \;\; ; \;\; \forall \theta, \tau \in \mathbb{R}^q \text{ and } \|\Lambda_\infty \theta\|_{\mathbb{R}^q} = \|\theta\|_{\mathbb{R}^q} ; \forall \theta \in \mathbb{R}^q$$

Hence, for $f \in L^2(0, +\infty; \mathbb{R}^n)$, there exists a unique θ_f in $\mathbb{R}^q$ such that

$$\Lambda_\infty \theta_f = -R_C^\infty f$$

and the control u_{θ_f} given by $u_{\theta_f} = (H_C^\infty)^* \theta_f$ verifies $R_C^\infty f + H_C^\infty u_{\theta_f} = 0$. Note also that it is optimal with $\left\| u_{\theta_f} \right\|_{L^2(0,T;\mathbb{R}^p)} = \|\theta_f\|_{\mathbb{R}^q}$. On the other hand, we have $\theta_f = -(\Lambda_\infty)^{-1} R_C^\infty f$, then

$$u_{\theta_f} = -(R_C^\infty)^* \Lambda_\infty^{-1} R_C^\infty f$$

Hence, if K is the linear operator defined by

$$K = -(H_C^\infty)^* (\Lambda_\infty)^{-1} R_C^\infty$$

the system (S_∞) augmented with (E_∞), is K-remediable asymptotically.

2) Now, if $(S_\infty) + (E_\infty)$ is K-remediable asymptotically, we have

$$ker[R_C^\infty + H_C^\infty K] = \mathcal{E}_\infty$$

then, for any $f \in L^2(0, +\infty; \mathbb{R}^n)$, we have

$$R_C^\infty f + H_C^\infty K f = 0$$

which may be written $R_C^\infty f + H_C^\infty u = 0$, with $u = K\, f \in L^2(0, +\infty; \mathbb{R}^p)$. Consequently $(S_\infty) + (E_\infty)$ is asymptotically remediable. □

We give hereafter an application to a two dimension system.

Example 4.2. *We consider, without loss of generality, a two dimension system*

$$\begin{cases} \dot{z}(t) &= Az(t) + Bu(t) + f(t) \; ; \; t > 0 \\ z(0) &= z_0 \end{cases}$$

augmented with the output equation :

$$y(t) = Cz(t) \; ; \; t > 0$$

where

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \;\; ; \;\; B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \;\; and \;\; C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in L^2(0,+\infty;\mathbb{R}^2)$ *and* $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in L^2(0,+\infty;\mathbb{R}^2)$. *Using the rank condition, the considered system is obviously remediable, and hence* K*-remediable with* $K = -(H_C^\infty)^*(\Lambda_\infty)^{-1}R_C^\infty$. *We have*

$$R_C^\infty f = \int_0^{+\infty} Ce^{At}f(t)dt = \int_0^{+\infty} \begin{bmatrix} 0 & e^{-t} \end{bmatrix} \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} dt = \int_0^{+\infty} f_2(t)e^{-2t}dt$$

Hence, if $f_2(t) = te^{-t}$, *then*

$$R_C^\infty f = \int_0^{+\infty} te^{-3t}dt = \frac{1}{3}$$

On another hand, the corresponding linear operator $\Lambda_\infty : \mathbb{R} \longrightarrow \mathbb{R}$ *is defined as follows:*

$$\begin{aligned} \Lambda_\infty\theta &= \int_0^{+\infty} Ce^{At}BB^{tr}e^{A^{tr}t}C^{tr}\theta dt \\ &= \int_0^{+\infty} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \theta dt \end{aligned}$$

It is easy to show that

$$\Lambda_\infty\theta = 25\,\theta \int_0^{+\infty} e^{-4t}dt = \frac{25\,\theta}{4}$$

Consequently Λ_∞^{-1} *is defined by* $\Lambda_\infty^{-1}\theta = \dfrac{4\,\theta}{25}$. *Hence, the control* u *is given by*

$$u = Kf = -(H_C^\infty)^*(\Lambda_\infty)^{-1}R_C^\infty f = -(H_C^\infty)^*\frac{4}{25} \times \frac{1}{3} = -(H_C^\infty)^*\frac{4}{75}$$

We obtain easily $(H_C^\infty)^*\theta = \begin{bmatrix} 3e^{-2t} \\ 4e^{-2t} \end{bmatrix}\theta$, *then*

$$u = -\frac{4}{75}\begin{bmatrix} 3e^{-2t} \\ 4e^{-2t} \end{bmatrix}$$

Consequently

$$H_C^\infty u = \int_0^{+\infty} Ce^{At}Bu(t)dt = -\frac{4}{75}\int_0^{+\infty} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3e^{-2t} \\ 4e^{-2t} \end{bmatrix} dt$$

We have

$$H_C^\infty u = -\frac{4}{75}\int_0^{+\infty} 25e^{-4t}dt = -\frac{1}{3}$$

Consequently,

$$R_C^\infty f + H_C^\infty u = \frac{1}{3} - \frac{1}{3} = 0$$

i.e. the control $u = Kf$ *compensates asymptotically the disturbance* f. *This result remain true for any disturbance* f. *One can also examine by the same the case of a finite time horizon. The results are similar.*

In this section concerning the asymptotic case, we obtain analogous properties and results to those given in the previous sections concerning a finite time horizon.

Note also that for a given operator $K : L^2(0,+\infty;\mathbb{R}^n) \to L^2(0,+\infty;\mathbb{R}^p)$, the system $(S_\infty)+(E_\infty)$ is K-remediable asymptotically on $F = ker[R_C^\infty + H_C^\infty K]$. Thus asymptotically, $(S_\infty) + (E_\infty)$ may be K-remediable on a subspace $F \subset L^2(0, +\infty; \mathbb{R}^n)$, without being K-remediable on the whole space $L^2(0, +\infty; \mathbb{R}^n)$. In the case of disturbances of the form $f(t) = Dv(t)$, with $D \in M_{n,r}(\mathbb{R})$ and $v \in L^2(0, +\infty; \mathbb{R}^r)$, the definitions and the results are analogous.

5 Conclusion

In this paper, we first studied the possibility of finite time compensation of any disturbance f, or a class of disturbances, using directly feedback controls $u = K\ f$, where K is a linear operator. Appropriate definitions, characterization results and illustrative examples are given. The relationship with the remediability and the existence of such operators K is also examined and an extension to the asymptotic case is considered. The considered approach which depend on the matrices A, B and C, includes the case where the disturbances f may be unknown.

In fact, for distributed parameter (infinite dimension) systems in finite time or asymptotic cases, there are two notions: weak and exact remediability [Afifi, L. et al ([1], [2], [3], [4], [6])]. These notions are equivalent in the case where the observation is given by a finite number of sensors and for lumped systems.

Competing Interests

The authors declare that no competing interests exist.

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