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Problem of Determining the Two-Dimensional Absorption Coefficient in a Hyperbolic-Type Equation

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Abstract

The problem of determining the hyperbolic equation coefficient on two variables is considered. Some additional information is given by the trace of the direct problem solution on the hyperplane $x = 0$. The theorems of local solvability and stability of the solution of the inverse problem are proved.

Keywords: Inverse Problem, Hyperbolic Equation, Delta Function, Local Solvability

1. Statement of the Problem and the Main Results

We consider the generalized Cauchy problem

$$u_{tt} - u_{xx} - b(x,t)u_t = \delta(x, t-s), \quad (x,t) \in R^2, \ s > 0, \qquad u\big|_{t<0} \equiv 0, \tag{1}$$

where $\delta(x,t)$ is the two-dimensional Dirac delta function, $b(x,t)$ is a continuous function, s is a problem parameter, and $u(x,t,s)$. We pose the inverse problem as follows: it is required to find absorption coefficient $b(x,t)$ if the values of the solution for are known, *i.e.*, if the function

$$u(0,t,s) = f(t,s), \ t > 0, \ s > 0. \tag{2}$$

Definition. *A function $b(x,t)$ such that the solution of problem* (1) *corresponding to this function satisfies relation* (2) *is called a solution of inverse problem* (1), (2).

The inverse problem posed in this paper is two-dimensional. For the case where $b(x,t) = b(x)$ the solvability problems for different statements of problems close to (1), (2) were studied in [1] (Chapter 2) and [2] (Chapter 1). The solvability problems for multidimensional inverse problems were considered in [2] (Chapter 3), [3,4], where the local existence theorems were proved in the class of functions smooth one of the variables and analytic in the other variables. In [5], the problems of stability and global uniqueness were investigated for inverse problem of determining the nonstationary potential in hyperbolic-type equation. In this paper, we prove the local solvability theorem and stability of the solution of the inverse problem (1), (2).

Let

$$Q_T := \{(t,s) \,|\, 0 \le s \le t \le T\},$$

$$\Omega_T := \{(x,t) \,|\, 0 \le |x| \le t \le T - |x|\}, \ T > 0,$$

$C_t^1(Q_T)$ is the class of function continuous in s, continuously differentiable in t, and defined on Q_T. We let B denote the set of function $b(x,t)$ such that

$$b(x,t) \in C(\Omega_T), \ b(-x,t) = b(x,t).$$

Theorem 1. *If at a $T > 0$ $f(t,s) \in C^1(Q_T)$ and the condition*

$$f(s+0,s) = \frac{1}{2} \tag{3}$$

is met, then for all $T \in (0, T\_0)$, $T_0 = (1/40)\alpha_0$, $\alpha_0 = 4\|f_t'(t,s)\|_{C(Q_T)}$ the solution to the inverse problems (1), (2) *in the class of function $b(x,t) \in B$ exists and is unique.*

Theorem 2. *Let the conditions in Theorem* 1 *hold for the functions $f_k(t,s)$, $k = 1,2$, and let $b_k(x,t)$, $k = 1,2$, be the solutions to the inverse problems with the data $f_k(t,s)$, $k = 1,2$, respectively. Then the following estimate is valid for $T \in (0, T_0)$, ((T_0) is defined in the same way as in proof of the Theorem* 1)

$$\|b_1(x,t) - b_2(x,t)\|_{C(\Omega_T)} \le \frac{4}{1-\rho}\|f_1(t,s) - f_2(t,s)\|_{C_t^1(Q_T)}, \tag{4}$$

where $\rho = \dfrac{T}{T_0}$.

2. Construction of a System Integral Equations for Equivalent Inverse Problems

We represent the solution of problem (1) as

$$u(x,t,s) = \frac{1}{2}\theta(t-s-|x|) + v(x,t,s). \quad (5)$$

where $\theta(t) = 1$ for $t \geq 0$, $\theta(t) = 0$, for $t < 0$, $v(x, t, s)$ is a some regular function.

We substitute the Expression (5) in (1), take into account that $\theta(t-s-|x|)/2$ satisfies (in the generalized sense) the equation $u_{tt} - u_{xx} = \delta(x)\delta'(t-s)$, and obtain the problem for the function v:

$$v_{tt} - v_{xx} = b(x,t)\left[\frac{1}{2}\delta(t-s-|x|) + v_t(x,t,s)\right], \quad (x,t)\in R^2,\ s>0, \quad v\big|_{t,0} \equiv 0. \quad (6)$$

It follows from the d'Alembert formula that the solution of problem (6) satisfies the integral equation

$$v(x,t,s) = \frac{1}{2}\iint\limits_{\Delta(x,t)} b(\xi,\tau)\ \left[\frac{1}{2}\delta(\tau-s-|\xi|) + v_t(\xi,\tau,s)\right]\cdot d\xi d\tau,\ (x,t)\in R^2,\ s>0, \quad (7)$$

where $\Delta(x,t) = \{(\xi,\tau) | 0 \leq \tau \leq t - |x-\xi|,\ x-t \leq \xi \leq x+t\}$. We use the properties of the δ - function and easily obtain the relation in a different form:

$$v(x,t,s) = \frac{1}{4}\int\limits_{\frac{x+(t-s)}{2}}^{\frac{x-(t-s)}{2}} b(\xi, s+|\xi|)d\xi + \frac{1}{2}\iint\limits_{Y(x,t,s)} b(\xi,\tau)v_t(\xi,\tau,s)d\tau d\xi, \quad t-s \geq |x|, \quad (8)$$

where the domain $Y(x,t,s)$ is defined by

$$Y(x,t,s) = \left\{(\xi,\tau) \middle| |\xi| + s \leq \tau \leq t - |x-\xi|, \frac{x-(t-s)}{2} \leq \xi \leq \frac{x+t-s}{2}, 0 \leq s \leq t,\ s = const\right\}.$$

By differentiating the equality (8), we obtain

$$v_t(x,t,s) = \frac{1}{8}\left[b\left(\frac{x+t-s}{2}, \frac{x+t+s}{2}\right) + b\left(\frac{x-t+s}{2}, \frac{-x+t+s}{2}\right)\right]$$
$$\frac{1}{2}\int\limits_{\frac{x-(t-s)}{2}}^{\frac{x+(t-s)}{2}} b(\xi,\ t-|x-\xi|)v_t(\xi,\ t-|x-\xi|,\ s)d\xi,\ t-s \geq |x|. \quad (9)$$

It is obvious that $f(t,s) = u(0,t,s) = \dfrac{1}{2} + v(0,t,s)$ for $t \geq 0$. Moreover, the function $f(t,s)$ be must satisfy the condition (9).

We set $x = 0$ in the equality (9), use the fact that the function $b(x,t)$ is even in x, and obtain the relation

$$f_t(t,s) = \frac{1}{4}b\left(\frac{t-s}{2}, \frac{t+s}{2}\right) + \int\limits_{-\frac{t-s}{2}}^{\frac{t-s}{2}} b(\xi, t-\xi)v_t(\xi, t-\xi, s)d\xi, \quad (t,s)\in Q_T.$$

We rewrite this equality, replacing $(t-s)/2$ with $|x|$ and $(t+s)/2$ with *t*, and solve it for $b(x,t)$. We obtain

$$b(x,t) = 4f_t'(t+|x|, t-|x|) - 4\int\limits_{-|x|}^{|x|} b(\xi, t+|x|-\xi)\cdot v_t(\xi, t+|x|-\xi, t-|x|)d\xi, \quad t \geq |x|. \quad (10)$$

Let

$$Y_T = \{(x,t,s) | |x| + s \leq t \leq T - |x|, 0 \leq s \leq t \leq T\}$$

The domain Y_T in the space of the variables x, t, and s is a pyramid with the base Ω_t and vertex $(0, T, T/2)$. To find the value of the function *b* at (x,t), it is hence necessary to integrate $b(x,t)$ over the interval with the endpoints $(-|x|, t)$ and $(|x|, t)$ and to integrate the function $v_t(x,t,s)$ over the interval with the endpoints $(-|x|, t, t-|x|)$ and $(|x|, t, t-|x|)$, which belong to the domain Y_T.

One can rewrite the system of Equations (9) and (10) in the nonlinear operator form,

$$\psi = \mathrm{A}\psi, \quad (11)$$

where

$$\psi = \begin{bmatrix} \psi_1(x,t,s) \\ \psi_2(x,t) \end{bmatrix} = \begin{bmatrix} v_t(x,t,s) - \frac{1}{8}\left[b\left(\frac{x+t-s}{2}, \frac{x+t+s}{2}\right) + b\left(\frac{x-t+s}{2}, \frac{-x+t+s}{2}\right)\right] \\ b(x,t) \end{bmatrix}$$

The operator A is defined on the set of functions $\psi \in C[Y_T]$ and, according to (9), (10), has the form

$$A = (A_1,\ A_2),$$

where

$$A_1\psi = \frac{1}{2}\int_{\frac{x-(t-s)}{2}}^{\frac{x+(t-s)}{2}} \psi_2\left(\xi, t-|x-\xi|\right)\{\psi_1(\xi, t-|x-\xi|)$$
$$+\frac{1}{8}\left[\psi_2\left(\frac{\xi+t-|x-\xi|-s}{2}, \frac{\xi+t-|x-\xi|+s}{2}\right)\right.$$
$$\left.+\psi_2\left(\frac{\xi-t+|x-\xi|+s}{2}, \frac{-\xi+t-|x-\xi|+s}{2}\right)\right]\}d\xi,$$

$$A_2\psi = 4f_t'(t+|x|,\ t-|x|) - 4\int_{-|x|}^{|x|}\psi_2(\xi, t+|x|-\xi)\cdot$$
$$\left\{\psi_1(\xi, t+|x|-\xi, t-|x|) + \frac{1}{8}\left[\psi_2\left(|x|, t\right)\right.\right.$$
$$\left.\left. + \psi_2\left(\xi-|x|, t-\xi\right)\right]\right\}\mathrm{d}\xi.$$

At fulfillment of the condition (3) the inverse problem (1), (2) is equivalent to the operator Equation (11).

3. Proofs of the Theorems

Define

$$\|\psi\|_T = \max\left(\|\psi_1\|_{C(Y_T)},\ \ \|\psi_2\|_{C(\Omega_T)}\right).$$

Let s be the set of $\psi \in C(Y_T)(\Omega_T \subset Y_T)$ that satisfy the following conditions:

$$\|\psi - \psi^0\|_T \le \|\psi^0\|_T,$$

where $\psi^0 = (\psi_{01},\ \ \psi_{02}) = (0,\ 4f_t'(t+|x|, t-|x|)$. It is obviously, that $\|\psi^0\|_T \le 4\|f_t'(t,s)\|_{C(Q_T)} = \alpha_0\ (Q_T \subset Y_T)$. Now we can show that if T is small enough, A is a contraction mapping operator in S. The local theorem of existence and uniqueness then follows immediately from the contraction mapping principle. First let us prove that A has the first property of a contraction mapping operator, *i.e.*, if $\psi \in S$, then $A\psi \in S$ when T is small enough. Let $\psi \in S$. It is then easy to see that

$$\|\psi\|_T \le \|\psi - \psi^0\|_T + \|\psi^0\|_T \le 2\alpha_0.$$

Furthermore, one has

$$|A_1\psi - \psi_{01}| \le \frac{1}{2}\int_{\frac{x-(t+s)}{2}}^{\frac{x+(t+s)}{2}} \left|\psi_2\left(\xi, t-|x-\xi|\right)\right|$$
$$\times\left\{\left|\psi\left(\xi, t-|x-\xi|, s\right)\right| + \frac{1}{8}\left[\left|\psi_2\left(\frac{\xi+t-|x-\xi|-s}{2},\right.\right.\right.\right.$$
$$\left.\left.\frac{\xi+t-|x-\xi|+s}{2}\right)\right| + \left|\psi_2\left(\frac{\xi-t+|x-\xi|+s}{2},\right.\right.$$
$$\left.\left.\left.\left.\frac{-\xi+t-|x-\xi|+s}{2}\right)\right|\right]\right\}d\xi \le \frac{5T}{8}\alpha_0\|\psi^0\|_T^2,$$

$$|A_2\psi - \psi_{02}| \le 4\int_{-|x|}^{|x|}\left|\psi_2\left(\xi б, t+|x|-\xi\right)\right|$$
$$\times\left\{\left|\psi_1\left(\xi, t+|x|-\xi, t-|x|\right)\right| + \frac{1}{8}\left[\left|\psi_2\left(|x|, t\right)\right|\right.\right.$$
$$\left.\left.+\left|\psi_2\left(\xi-|x|, t-\xi\right)\right|\right]\right\}d\xi \le 10T\alpha_0\|\psi^0\|_T^2.$$

Therefore, if $T^* = 1/10\alpha_0$, then for $T \in (0, T_0)$ the operator A satisfies the condition $A\psi \in S$. Consider next the second property of contraction mapping operator for A *i.e.*, if $\psi^{(1)} \in S$, $\psi^{(2)} \in S$, then $\|A\psi^{(1)} - A\psi^{(1)}\| \le \rho\|\psi^{(1)} - \psi^{(1)}\|$ with $\rho < 1$, when T is small enough. Let $\psi^{(1)} \in S, \psi^{(2)} \in S.$ Then one has

$$|A_1\psi^{(1)} - A_1\psi^{(2)}| \le \frac{1}{2}\left|\int_{\frac{x-(t+s)}{2}}^{\frac{x+(t+s)}{2}}\left\{\left(\psi_2^{(1)} - \psi_2^{(2)}\right)(\xi, t-|x-\xi|)\right.\right.$$
$$\left\{\psi_2^{(1)}(\xi, t-|x-\xi|, s) + \frac{1}{8}\left[\psi^{(1)}{}_2\left(\frac{\xi+t-|x-\xi|-s}{2},\right.\right.\right.$$
$$\left.\frac{\xi+t-|x-\xi|+s}{2}\right) + \psi^{(1)}{}_2\left(\frac{\xi-t+|x-\xi|+s}{2},\right.$$
$$\left.\left.\left.\frac{-\xi+t-|x-\xi|+s}{2}\right)\right]\right\} + \psi_2^{(2)}(\xi, t-|x-\xi|, s)$$
$$\times\left\{\left(\psi_1^{(1)} - \psi_1^{(2)}\right)(\xi, t-|x-\xi|, s) + \frac{1}{8}\left[\left(\psi_2^{(1)} - \psi_2^{(2)}\right)\right.\right.$$
$$\left(\frac{\xi+t-|x-\xi|-s}{2}, \frac{\xi+t-|x-\xi|+s}{2}\right)$$

$$+\left(\psi_2^{(1)}-\psi_2^{(2)}\right)$$

$$\left(\frac{\xi-t+|x-\xi|+s}{2},\frac{-\xi+t-|x-\xi|+s}{2}\right)\Big]\Big]\Big\}$$

$$d\xi\Big|\le\frac{5T}{2}\alpha_0\left\|\psi^{(1)}-\psi^{(2)}\right\|_T,$$

$$\left|A_2\psi^{(1)}-A_2\psi^{(2)}\right|\le 4\left|\int_{-|x|}^{|x|}\left\{\left(\psi_2^{(1)}-\psi_2^{(2)}\right)\right.\right.$$

$$\left(\xi,t+|x|-\xi\right)\left\{\psi_1^{(1)}\left(\xi,t+|x|-\xi,t-|x|\right)\right.$$

$$\left.+\frac{1}{8}\left[\psi_2^{(1)}\left(|x|,t\right)+\psi_2^{(1)}\left(\xi-|x|,t-\xi\right)\right]\right\}$$

$$+\psi_2^{(2)}\left(\xi,t+|x|-\xi\right)\times\left\{\left(\psi_1^{(1)}-\psi_1^{(2)}\right)\right.$$

$$\left(\xi,t+|x|-\xi,t-|x|\right)+\frac{1}{8}\left[\left(\psi_2^{(1)}-\psi_2^{(1)}\right)\right.$$

$$\left.\left.\left(|x|,t\right)+\left(\psi_2^{(1)}-\psi_2^{(1)}\right)\left(\xi-|x|,t-\xi\right)\right]\right\}\Big\}$$

$$d\xi\Big|\le 40T\alpha_0\left\|\psi^{(1)}-\psi^{(2)}\right\|_T.$$

It follows from the preceding estimates that if $T_0=1/40\,\alpha_0$, then for $T\in\left(0,T_0\right)$ the operator A is a contraction operator with $\rho=T/T_0$ on the set *S*. Therefore, the Equation (11) has a unique solution which belongs to *S* according to the contraction mapping principle. The solution is the limit of the sequence $\psi^{[n]}$, $n=0,1,2,...,$ where $\psi^{[0]}=\psi(0)$, $\psi^{[n+1]}=A\,\psi^{[n]}$, and the series

$$\psi^{[0]}+\sum_{n=0}^{\infty}\left(\psi^{[n+1]}-\psi^{[n]}\right)$$

converges not slower than the series

$$\left\|\psi^{[0]}\right\|_T+\sum_{n=0}^{\infty}\rho^n\left\|\psi^{[1]}-\psi^{[0]}\right\|_T$$

We now prove Theorem 2. Since the conditions Theorem 1 hold, the solution belong to the set *S* and $\|\psi_i\|_T\le 2\alpha_0$, $i=1,2$. Let $\psi^{(k)}$, $k=1,2$ be vector functions which are the solution of the Equation (11) with the data $f_k\left(t,s\right)$, $k=1,2,$ respectively, *i.e.*,

$$\psi^{(k)}=A\,\psi^{(k)}$$

From the previous results in the proof of Theorem 1, it follows that

$$\left|\psi_1^{(k)}-\psi_2^{(k)}\left(x,t,s\right)\right|\le 4\left\|f_1\left(t,s\right)-f_2\left(t,s\right)\right\|_{C_t^1(Q_T)}$$

$$+40T\alpha_0\left\|\psi_1-\psi_2\right\|_T,\ k=1,2.$$

Therefore, one has

$$\left\|\psi_1-\psi_2\right\|_T\le 4\left\|f_1\left(t,s\right)-f_2\left(t,s\right)\right\|_{C_t^1\left(Q_T\right)}+\rho\left\|\psi_1-\psi_2\right\|_T$$

The last inequality gives

$$\left\|\psi_1-\psi_2\right\|_T\le\frac{4}{1-\rho}\left\|f_1\left(t,s\right)-f_2\left(t,s\right)\right\|_{C_t^1\left(Q_T\right)}\qquad(12)$$

The stability estimate (4) follows from the inequality (12).

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