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Mathematical Analysis of a Class of Surface-Tension Driven Flows

Yuh-Yih Chen[1], Jenn-Tsann Lin[1*] and Hsiu-Chuan Wei[1]

[1] *Department of Applied Mathematics, Feng Chia University, Seatwen, Taichung 40724, Taiwan.*

Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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ABSTRACT

In this paper, a two-point boundary value problem arising from a class of surface-tension driven flows is considered. The existence properties of solutions are established, and all possible solutions are classified using mathematical analysis. The problem possesses unique or multiple solutions depending on parameter values. Bifurcation diagrams are computed to verify the results obtained by mathematical analysis.

Keywords: Similarity solutions; classification; shooting scheme; homogeneity method.

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1 INTRODUCTION

The Navier-Stokes equations are the basic equations governing the motion of viscous fluid. Since these equations are necessarily nonlinear and complicated when applied to realistic problems, analytical results are often restricted to particular models with special properties. However, in some certain flows, the Navier-Stokes equations are reduced to nonlinear

*Corresponding author: E-mail: jtlin@math.fcu.edu.tw

ordinary differential equations through a similarity transform for studying the solution properties [1, 2, 3, 4, 5]. In this paper, a two point boundary value problem(TPBVP)

$$f''' + Q(Aff'' - f'^2) = \beta, \qquad Q, \beta \in \Re; A \geq 1, \tag{1.1}$$

is studied. The given problem arises from a similarity reduction of boundary layer approximation of Navier-Stokes system in a microgravity environment [6]. The Navier-Stokes system was applied to describe the steady state for the distributions of velocity in a low Prandtl(Pr) number fluid in a slot with an insulated bottom. Here Q is related to the Prandtl number, β is an integrable constant, $f(y)$ is related to the stream function, and $y = 1$ denotes the insulated bottom of the slot. For the derivation of equation (1.1) and (1.2) we refer to [7, 8]. Numerical solutions of the TPBVP for $A = 1$ and $A = 2$ were studied by Hwang et al. [8] using a multiple shooting code BVPSOL. Hwang et al. also proved the existence properties for a portion of the solutions for $A = 1$ and $A = 2$ using mathematical analysis. The rest of the existence properties of the solutions for $A = 1$ and $A = 2$ were proved by Hwang and Wang [9]. It is our purpose to study the TPBVP for $A \geq 1$. To provide details, mathematical analysis of the existence properties of the solutions for $A \geq 1$ is given in Sec. 2. Numerical computation of bifurcation diagrams and discussion is given in Sec. 3. Sec. 4 provides a brief conclusion.

subject to the boundary conditions

$$f(0) = f(1) = f''(0) + 1 = f'(1) = 0 \tag{1.2}$$

2 MATHEMATICAL ANALYSIS OF EXISTENCE PROPERTIES OF SOLUTIONS

Note that for each A, if $Q = 0$, the TPBVP has a unique solution $f(\eta) = \eta(1-\eta)^2/4$ for $\beta = 3/2$. Therefore, $Q \neq 0$ is assumed in our study. Let $y = b(1-\eta)$ and $g(y) = Qf(\eta)/b$ for $Q \neq 0$ and $b > 0$. The TPBVP is equivalent to

$$g''' + g'^2 - Agg'' = -Q\beta/b^4, \tag{2.1}$$

subject to the conditions

$$g(0) = g(b) = g'(0) = g''(b) + (Q/b^3) = 0. \tag{2.2}$$

Denote $g''(0)$ and $-Q\beta/b^4$ by α and B, respectively. By assuming values α and B, Eqs. (2.1) and (2.2) become the initial value problem:

$$g''' + g'^2 - Agg'' = B, \tag{2.3}$$

$$g(0) = g'(0) = g''(0) - \alpha = 0. \tag{2.4}$$

Suppose that the solution $g(y; \alpha, B, A)$ to Eqs. (2.3) and (2.4) meets the $y - axis$ at a positive value y^*. By setting $b = y^*$, the initial value problem in Eqs. (2.3) and (2.4) has a solution when $Q = -(y^*)^3 g''(y^*)$ and $\beta = -B(y^*)^4/Q$. Given $A \geq 0$, we denote $g(y; \alpha, B) = g(y; \alpha, B, A)$. $g(y; \alpha, B)$ can be extended to the maximal interval $[0, M)$, where $M = M(\alpha, B) \leq \infty$. In fact, g tends to ∞ or $-\infty$ as y approaches M if $M < \infty$. Therefore, the classification of positive zeros of g is given by (α, B) chosen from the following quadrants:

$$D_1 = \{(\alpha, B) \mid \alpha \geq 0, B > 0\},$$

$$D_2 = \{(\alpha, B) \mid \alpha < 0, B > 0\},$$

$$D_3 = \{(\alpha, B) \mid \alpha \leq 0, B < 0\},$$

and

$$D_4 = \{(\alpha, B) \mid \alpha > 0, B < 0\}.$$

We shall classify all possible solutions of Eqs. (1.1) and (1.2) by assuming values of α, B, A in Eqs. (2.1) and (2.3). It is clear that $g(y; 0, 0, A) = 0$ for all $A > 0$, $g(y; 0, B, 3/2) = \frac{B}{6}\eta^3$ for all $B \in \Re$, and $g(y; \alpha, 0, 2) = \frac{1}{2}\alpha\eta^2$ for all $\alpha \in \Re$. Thus, $(\alpha, B) \neq (0, 0)$, $(\alpha, A) \neq (0, \frac{3}{2})$, and $(B, A) \neq (0, 2)$ are assumed in the following discussions. Moreover, let $(0, M)$ be the corresponding maximal interval of $g(y; \alpha, B, A)$, where $M = M(\alpha, B, A)$. Note that g can only blow up to ∞ or $-\infty$ if $M < \infty$. The following expressions are used frequently in the mathematical analysis:

$$g''' = B - (g')^2 + Agg'', \quad (2.5)$$
$$g^{(iv)} = (A-2)g'g'' + Agg''', \quad (2.6)$$
$$g^{(v)} = (A-2)(g'')^2 + (2A-2)g'g''' + Agg^{(iv)}. \quad (2.7)$$

2.1 $A \geq 1$ and $B \leq 0$

Lemma 2.1. *For $A \geq 0$ and $B \leq 0$, $g''(y;\alpha,B,A)$ has at most one zero for all $\alpha \in \Re$.*

Proof. Assume that y_1 and y_2 are the first and second zero of g'', respectively. By (2.3), $g'''(y_i) = B - g'(y_i)^2 \leq 0$ for i=1,2. If the equality holds for i = 1 or 2, then $B = 0$, $g'(y_i) = 0$, and $g''(y_i) = 0$. Consider Eq. (2.3) together with the initial condition $g(y_i)$, $g'(y_i) = 0$, and $g''(y_i) = 0$. Then $g(y) \equiv g(y_i)$, $y \in [y_i, M)$ is the solution. In fact, the solution $g(y) \equiv g(y_i)$ can be extended to the maximal interval $[0, M)$. Therefore $g(y) \equiv 0$. This contradicts the assumption that $(\alpha, B) \neq (0,0)$. Therefore $g'''(y_i) = B - g'(y_i)^2 < 0$ for i=1,2. This implies that g'' has a zero in (y_1, y_2), which is a contradiction.

Theorem 2.2. *For $A > 0$, $B \leq 0$, and $\alpha \leq 0$, $g(y;\alpha,B,A) < 0$ on $(0,M)$.*

Proof. Since $g'''(0) = B \leq 0$, $g''(0) = \alpha \leq 0$, and $(\alpha, B) \neq (0,0)$, g'' is negative initially. Assume that g'' has a zero on $(0,M)$ and let $\bar{y}$ be the first positive zero. This implies that $g'''(\bar{y}) \geq 0$ and g' is negative on $(0,\bar{y})$, but $g'''(\bar{y}) = B - g'(\bar{y})^2 < 0$ is a contradiction. Therefore, $g'' < 0$ on $(0,M)$. This, together with the initial conditions $g'(0) = 0$ and $g(0) = 0$, gives the result $g(y;\alpha,B,A) < 0$ on $(0,M)$.

Lemma 2.3. *For $A \in [0,2)$, $B \leq 0$, and $\alpha > 0$, $g''(y;\alpha,B,A)$ has exactly one zero.*

Proof. Assume $g'' > 0$ on $(0,M)$ and then $g > 0$ and $g' > 0$ on $(0,M)$. Let $\mu(y) = \exp(-A\int_0^y g)$. We have $(\mu g''')' = (A-2)\mu g'g'' < 0$ and thus $g''' \leq B\exp(A\int_0^y g) \leq 0$. Thus, $g^{(iv)} < 0$ implying that g'' is concave downward on $(0,M)$ which contradicts to $g'' > 0$ on $(0,M)$. Hence, $g''(y;\alpha,B,A)$ has at least one zero. From Lemma 2.1, $g''(y;\alpha,B,A)$ has exactly one zero.

Theorem 2.4. *For $A \in [0,2)$, $\alpha > 0$, and $B \leq 0$, $g(y;\alpha,B,A)$ has exactly one zero.*

Proof. Let y_2 be the zero of g'' and assume that $g' > 0$ on $(0,M)$ which leads to $g > 0$ on $(0,M)$. By the proof of Lemma 2.3, $g'''(y) \leq B \leq 0$ on $(0,y_2)$. Thus, $g''' = B - g'^2 + Agg'' < 0$ on (y_2, M), and g' is concave downward on $(0,M)$. This contradicts to $g' > 0$ on $(0,M)$, and g' has exactly one positive zero. Similarly, g has exactly one zero.

Theorem 2.5. *For $A > 2$, $\alpha > 0$, and $B \leq 0$, $g(y;\alpha,B,A)$ has either one or no zero.*

Proof. It is easy to prove that if $g'' > 0$ on (0,M), then g has no zero, and if g'' has exactly one zero on $(0,M)$, then g has exactly one zero on $(0,M)$. By Theorem 2.2, $g(y;0,B,A) < 0$ on $(0,M)$ for all $B < 0$. By continuous dependence on the initial data, if α is sufficiently small, then g has exactly one zero.

2.2 $A \geq 1$ and $B > 0$

Lemma 2.6. *For $A > 2$ and $\alpha \geq 0$, $g^{(iv)}(y;\alpha,B,A) > 0$ on $(0,M)$.*

Proof. Assume that $B > 0$. From Eq. (2.4), we have $g'''(0) = B > 0$. Therefore, all of g, g', and g'' are increasing and positive initially. When $g^{(k)}(t) > 0$ for all $0 \leq k \leq 3$ and $A \geq 2$, we have $g^{(iv)}(t) > 0$. So, $g^{(k)}(t) > 0$ is increasing at t for all $0 \leq k \leq 3$. Therefore, $g^{(k)}(t) > 0$ on $(0,M)$ for all $0 \leq k \leq 4$ if $B > 0$ and $A \geq 2$. Now if $B = 0$, we may assume $\alpha > 0$. Since $g^{(iv)}(0) = 0$ and $g^{(v)}(0) > 0$ for $A > 2$, $g^{(iv)}$ is increasing and positive initially. Therefore, $g^{(k)}(t) > 0$ on $(0,M)$ for all $0 \leq k \leq 4$ by similar arguments as stated above.

Theorem 2.7. *For $A > 2$, $\alpha \geq 0$, $g(y;\alpha,B,A) > 0$ on $(0,M)$.*

Proof. Note that $g(t) = \frac{\alpha}{2}t^2$ is the solution for $A = 2$ and $B = 0$. This fact, together with Lemma 2.6, completes the proof of this theorem.

Lemma 2.8. *For $A > 2$, $\alpha \leq 0$, and $B > 0$, $g''(y;\alpha,B,A)$ has at most one zero.*

Proof. If $g''' > 0$ on $(0, M)$, the Lemma is clear. Suppose that g''' has a positive zero and let y_0 be the first zero of g'''. It follows that $g^{(iv)}(y_0) \leq 0$ from Eq. (2.6). Hence, $g'(y_0)g''(y_0) \leq 0$ if $A > 2$. Because $g''' > 0$ on $(0, y_0)$, $g''(y_0) > 0$ and $g'(y_0) \leq 0$. Now, we divide the proof into two cases. Case (i): $g'(y_0) = 0$. In this case, $g'''(y_0) = g^{(iv)}(y_0) = 0$ and $g^{(v)}(y_0) > 0$. Thus, $g''' > 0$ on $(y_0, y_0 + \delta)$ for some $\delta > 0$. Suppose that $g''' > 0$ on (y_0, y_1) and $g'''(y_1) = 0$. This implies that both g'' and g' are positive and increasing on (y_0, y_1). So $g^{(iv)}(y_0) > 0$. This is a contradiction. Therefore, y_0 is the unique zero of g''', and g'' has exactly one zero. Case (ii): $g'(y_0) < 0$. This leads that $g'(y_0)^2 < B$. Let $y_* \in (0, y_0)$ be the first zero of g''. Thus, g and g' are negative on $[y_*, y_0]$. This implies $g^{(iv)} \leq 0$ on (y_*, y_0). Now, we claim that $g'' > 0$ for all $y > y_0$. Assume that y^* is the second zero of g''. Since g'' and g''' cannot both be zero at y^*, we have $g'''(y^*) < 0$. In fact, $g''' \leq 0$ on $[y_0, y^*]$. Otherwise, there exists y_2 in (y_0, y^*) such that $g'''(y_2) = 0$ and $g^{(iv)}(y_2) > 0$. This implies $g'(y_2) > 0$. By similar arguments as in Case (i), $g'' > 0$ for $y > y_0$. This contradicts the assumption that g'' has a second zero at y^*. Let $y_3 > y_0$ be a zero of g satisfying $g < 0$ on $[y_0, y_3)$. Furthermore, let $\bar{y} = \min\{y_3, y^*\}$. Thus $g'''(\bar{y}) \leq 0$ and $g^{(iv)} > [A-2]g'g''$ on $[y_0, \bar{y}]$. Then,

$$\begin{aligned} \int_{y_0}^{\bar{y}} g^{(iv)}(y)dy &> \int_{y_0}^{\bar{y}} [A-2]g'(y)g''(y)dy, \\ g'''(\bar{y}) &> \frac{A-2}{2}(g'(\bar{y}))^2 - \frac{A-2}{2}(g'(y_0))^2. \end{aligned}$$

Next,

$$\begin{aligned} g'''(\bar{y}) - \frac{A-2}{2}[B + Ag(\bar{y})g''(\bar{y}) - g'''(\bar{y})] &= g'''(\bar{y}) - \frac{A-2}{2}(g'(\bar{y}))^2 \\ &\geq -\frac{A-2}{2}g'^2(y_0) \\ &> -\frac{A-2}{2}B. \end{aligned}$$

Thus, $g''' - (A-2)gg'' > 0$ at $y = \bar{y}$. It contradicts the sign of $g'''(\bar{y})$. Thus, $g'' > 0$ for all $y > y_0$ and therefore, the proof is complete.

The following theorem is obtained immediately.

Theorem 2.9. *For $A > 2$, $\alpha < 0$, and $B > 0$, $g(y; \alpha, B, A)$ has at most one zero.*

For the mathematical analysis of the rest of the cases, a notation containing the sign of $g^{(k)}$, where $k = 0, 1, \cdots, 5$, is defined with $(\text{sign}\, g, \text{sign}\, g', \cdots, \text{sign}\, g^{(v)})$. We use “+”“-”,“0”,“+0”,“-0”, and “*” to indicate positive, negative, zero, positive or zero, negative or zero, and indeterminate or unimportant, respectively. For example, $(+, -, 0, +0, -0, *)$ at y means that $g(y) > 0$, $g'(y) < 0$, $g''(y) = 0$, $g'''(y) \geq 0$, $g^{(iv)}(y) \leq 0$, and the sign of $g^{(v)}(y)$ is indeterminate or it does not affect the result of the analysis.

Lemma 2.10. *For $A \in (1, 2)$, $\alpha \geq 0$, and $B > 0$, $g''(y; \alpha, B, A)$ has at most one zero.*

Proof. From the initial condition Eq. (2.4), we have $(+, +, +, +, *, *)$ on $(0, \delta)$ for some $\delta > 0$. Let y_* be the first zero of g'', then we have $(+, +, 0, -, -, -)$ at y_* because g'' and g''' cannot be zero simultaneously. Consequently, we have $(+, +, -, -, -, -)$ on $(y_*, y_* + \delta_1)$ for some $\delta_1 > 0$. Suppose that $g^{(iv)}(y_1) = 0$ for some $y_1 > y_*$ and $g^{(iv)}(y) < 0$ for $y \in (y_*, y_1)$. Because $g''(y) < 0$ and $g'''(y) < 0$ for $y \in (y_*, y_1]$, there are three possible cases of g and g' values at y_1: (i) $(-0, -, -, -, 0, +0)$, (ii) $(+, -, -, -, 0, +0)$, or (iii) $(+, +0, -, -, 0, +0)$.

For case (i), if $g'''(y_2) = 0$ for some $y_2 > y_1$ and $g'''(y) < 0$ for $y \in (y_1, y_2)$, then $g^{(iv)}(y_2) \geq 0$. Now, $g'''(y_2) = 0$ and $g^{(iv)}(y_2) = 0$ imply $g'(y_2) = 0$ or $g''(y_2) = 0$, which cannot hold in this case. Therefore, $g^{(iv)}(y_2) > 0$. Thus, we have $(-, -, -, 0, +, *)$ at y_2. However, this contradicts the sign of $g^{(iv)}(y_2)$ determined by Eq. (2.6).

For, case (ii), $g^{(iv)}(y_1) < 0$ from Eq. (2.6). This contradicts with the assumption $g^{(iv)}(y_1) = 0$. For case (iii), $g^{(v)}(y_1) < 0$ from Eq. (2.7). This contradicts with the sign of $g^{(v)}(y_1)$ in this case.

The above three cases give the conclusion that $g''' < 0$ for $y \in (y_*, M)$. Therefore, $g'' < 0$ for $y \in (y_*, M)$. Lemma 2.10 proves the following theorem.

Theorem 2.11. *For $A \in (1, 2)$, $\alpha \geq 0$ and $B > 0$, $g(y; \alpha, B, A)$ has at most one zero.*

Theorem 2.12. *For $A \in (1, 2)$, $\alpha < 0$ and $B > 0$, $g(y; \alpha, B, A)$ has at most two zeros.*

Proof. From the initial condition in Eq. (2.4), we have $(-, -, -, +, -, -)$ on $(0, \delta)$ for some $\delta > 0$. We let $y_1 > 0$ such that there are two possible cases: (i) y_1 is the first zero of g''', and g, g', and g'' do not change their sign in $(0, y_1]$. (ii) y_1 is the first zero of g'', and g, g', and g''' do not change their sign in $(0, y_1]$.

For case (i), $g^{(iv)}(y_1) < 0$ from Eq. (2.6). From case (i) in the proof of Lemma 2.10, $g'''(y) < 0$ for $y \in (y_1, M)$ and $g(y; \alpha, B, A)$ has no zero on $(0, M)$.

For case (ii), $g^{(iv)}(y_1) < 0$ and we have $(-, -, +, +, -, *)$ on $(y_1, y_1 + \delta_1)$ for some $\delta_1 > 0$. We let $y_2 > y_1$ such that there are two possible cases: (a) $g'''(y_2) = 0$, $g'''(y) > 0$ for $y \in (y_1, y_2)$, and we have $(-, -, +, +0, *, *)$ on $(y_1, y_2]$. (b) $g'(y_2) = 0$, $g'(y) < 0$ for $y \in (y_1, y_2)$, and we have $(-, -0, +, +, *, *)$ on $(y_1, y_2]$.

Case (a) is impossible because $g^{(iv)}(y_2) > 0$ from Eq. (2.6).

For case (b), we have $(-, +, +, +, -, *)$ on $(y_2, y_2 + \delta_2)$ for some $\delta_2 > 0$. We let $y_3 > y_2$ such that there are two possible cases: (1) $g(y_3) = 0$, $g(y) < 0$ for $y \in (y_2, y_3)$, and we have $(-0, +, +, +, *, *)$ on $(y_2, y_3]$. (2) $g'''(y_3) = 0$, $g'''(y) > 0$ for $y \in (y_2, y_3)$, and we have $(-, +, +, +0, *, *)$ on $(y_2, y_3]$.

For case (1), $y^{(iv)}(y_3) < 0$ from Eq. (2.6), and we have $(+, +, +, +, -, *)$ on $(y_3, y_3 + \delta_3)$ for some $\delta_3 > 0$. From the proof of Lemma 2.10, g has at most one zero in (y_3, M), and thus, g has at most two zeros in $(0, M)$.

For case (2), $y^{(iv)}(y_3) < 0$, and we have $(-, +, +, -, -, *)$ on $(y_3, y_3 + \delta_4)$ for some $\delta_4 > 0$. We let $y_4 > y_3$ such that there are three possible cases: (A) $g'''(y_4) = 0$, $g'''(y) < 0$ for $y \in (y_3, y_4)$, and we have $(-, +, +, -0, *, *)$ on $(y_3, y_4]$. (B) $g(y_4) = 0$, $g(y) < 0$ for $y \in (y_3, y_4)$, and we have $(-0, +, +, -, *, *)$ on $(y_3, y_4]$. (C) $g''(y_4) = 0$, $g''(y) > 0$ for $y \in (y_3, y_4)$, and we have $(-, +, +0, -, *, *)$ on $(y_3, y_4]$.

Case (A) is impossible because $g^{(iv)}(y_4) < 0$ from Eq. (2.6).

For case (B), we have $(+, +, +, -, -, *)$ on $(y_4, y_4 + \delta_5)$ for some $\delta_5 > 0$.

We may apply the proof of Lemma 2.3 and Theorem 2.4, and g has exactly one zero in (y_4, M). Therefore, g has exactly two zeros in $(0, M)$.

For case (C), $y^{(iv)}(y_4) > 0$, from Eq. (2.6), implies that there is $\bar{y}$, where $y_3 < \bar{y} < y_4$, such that $y^{(iv)}(\bar{y}) = 0$ and $y^{(iv)}(y) > 0$ for $y \in (\bar{y}, y_4]$. However, $g^{(v)}(\bar{y}) < 0$ from Eq. (2.7). Therefore, case (C) is impossible.

From all the cases discussed above, we have concluded that $g(y; \alpha, B, A)$ has at most two zeros.

From the above lemmas and theorems and the cases studied by Hwang et al. [8] for $A = 1$ and $A = 2$, the existence properties of solutions for $A \geq 1$ are summarized as follows:

(i) For $B \leq 0$, $\alpha \leq 0$, and $A \geq 1$, g has no zero.

(ii) For $B \leq 0$ and $\alpha > 0$, g has one zero if $A \in [1, 2)$, and g has at most one zero if $A \geq 2$.

(iii) For $B > 0$ and $\alpha < 0$, g has at most two zeros if $A \in [1, 2)$, and g has at most one zero if $A \geq 2$.

(iv) For $B > 0$ and $\alpha \geq 0$, g has at most one zero if $A \in [1, 2)$, and g has no zero if $A \geq 2$.

3 NUMERICAL SIMULATIONS AND DISCUSSION

When the solutions to a differential equation satisfy a homogeneity property, an efficient numerical approach can be constructed to reduce the computational effort [7, 10, 11]. In this section, a homogeneity property of the solutions to Eqs. (2.1) and (2.2) will be established, and a numerical approach will be developed for computing the bifurcation diagrams using Q and β as the bifurcation parameters.

3.1 Preliminaries

The solution $g(y;\alpha,B,A)$ satisfies the following property.

Proposition 3.1. $g(y;\alpha,B)=\lambda g(\lambda y;\alpha/\lambda^3,B/\lambda^4)$*, for* $\lambda>0$.

Proof. Define $h(t)=g(t/\lambda;\alpha,B)/\lambda$. Let $y=t/\lambda$. Then, $\lambda h(\lambda y)=g(y;\alpha,B)$, and $\lambda^{i+1}h^{(i)}(t)=g^{(i)}(y;\alpha,B)$, where $i=1,2,3$. We also have $h(0)=0,\ h'(0)=0,\ h''(0)=\alpha/\lambda^3$. Therefore, $h(t)$ satisfies the following problem:

$$h'''+h'^2-Ahh''=B/\lambda^4$$
$$h(0)=h'(0)=h''(0)-\alpha/\lambda^3=0.$$

Thus, $h(t)=g(t;\alpha/\lambda^3,B/\lambda^4)=g(\lambda y;\alpha/\lambda^3,B/\lambda^4)$. So, $g(y;\alpha,B)=\lambda g(\lambda y;\alpha/\lambda^3,B/\lambda^4)$.

Now, let $y(\alpha,B)$ be the positive zero, if there is any, of $g(y;\alpha,B)$. Then $y(\alpha,B)$, Q and β satisfy the following homogeneity property.

Proposition 3.2. *For* $\lambda>0$*,*

$$y(\alpha,B)=y(\alpha/\lambda^3,B/\lambda^4)/\lambda,$$
$$Q(\alpha,B)=Q(\alpha/\lambda^3,B/\lambda^4),$$

and

$$\beta(\alpha,B)=\beta(\alpha/\lambda^3,B/\lambda^4).$$

Proof. From Proposition 3.1, $g(y;\alpha,B)=0$ implies $g(\lambda y;\alpha/\lambda^3,B/\lambda^4)=0$. So, $y(\alpha/\lambda^3,B/\lambda^4)=\lambda y(\alpha,B)$. For the second equation,

$$Q(\alpha,B)=\frac{-By^4(\alpha,B)}{\beta}=\frac{-By^4(\alpha/\lambda^3,B/\lambda^4)}{\beta\lambda^4}=Q(\alpha/\lambda^3,B/\lambda^4).$$

For the third equation,

$$\beta(\alpha,B)=\frac{-By(\alpha,B)}{Q(\alpha,B)}=\frac{-By(\alpha/\lambda^3,B/\lambda^4)}{\lambda Q(\alpha/\lambda^3,B/\lambda^4)}=\beta(\alpha/\lambda^3,B/\lambda^4).$$

From the homogeneity property in Proposition 3.2, we have $Q(\alpha,B)=Q(\alpha/\lambda^3,B/\lambda^4)=Q(1,B/\alpha^{4/3})$, $\beta(\alpha,B)=\beta(1,B/\alpha^{4/3})$, and $y(\alpha,B)=y(1,B/\alpha^{4/3})/\lambda$, if we take $\lambda=\alpha^{1/3}$. This implies that every point (α,B) on the curve $B=k\alpha^{4/3}$, where k is a constant, corresponds to the same point on the $Q-\beta$ plane. Also, $g(y;\alpha,B,A)$ has the same number of positive zeros on the curve $B=k\alpha^{4/3}$. This property allows us to design a one-dimensional computational domain for the computation of solutions on the $\alpha-B$ plane.

To locate the possible zeros of $g(\eta;\alpha,B,A)$, an initial value problem code, SDRIV2 [12], is employed with (α,B) chosen along a simple closed curve around the origin in the $\alpha-B$ plane for every A. That is, we may pick the parameter (α,B) along the curve $|\alpha|+|B|=1$.

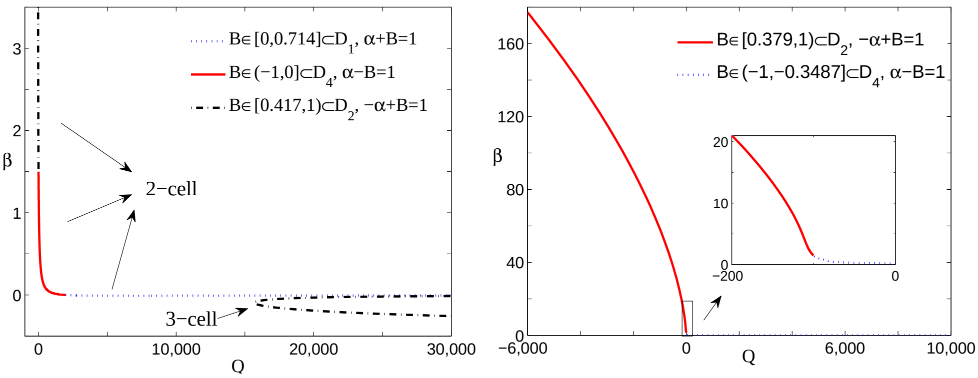

Fig. 1. Bifurcation diagram for (a) $A = 1.5$ and (b) A = 4.

When g has multiple zeros, Eqs. (1.1) and (1.2) have multiple solutions corresponding to different (Q, β) values. For example, let y_1 and y_2, where $0 < y_1 < y_2$, be two zeros of g. Under the transformation given in Sec. 2, $f(\eta) = y_i g(y)/Q_i$, is a solution to Eqs. (1.1) and (1.2) with parameter values $Q_i = -y_i^3 g''(y_i)$ and $\beta_i = -B(y_i)^4/Q_i$, $i = 1, 2$.

A solution f of Eqs. (1.1) and (1.2) is called a two-cell solution if f' has exactly one zero in $(0, 1)$, and it is called a three-cell solution if f' has exactly two zeros in $(0, 1)$ [7, 8, 6]. Recall that the TPBVP, Eqs. (1.1) and (1.2), arises from surface-tension flows in a slot with an insulated bottom. A two-cell or three-cell solution corresponds to a two-cell or three-cell flow. Because $g(y) = Qf(\eta)/b$ for $Q \neq 0$ and $b > 0$, where $y = b(1-\eta)$, f' and g' have the same number of zeros. Therefore, the number of zeros of g' can be used to determine if a solution is a two-cell or three-cell solution.

3.2 Numerical Simulations

In our first numerical example, we let $A = 1.5$, and Fig. 1(a) shows the bifurcation diagram for $(Q, \beta) \in [-1000, 30000] \times [0.5, 3.5]$. Theorem 2.2 shows that g has no zero when $(\alpha, B) \in D_3$. Along the curve $|\alpha| + |B| = 1$, the zeros of g are computed for $(\alpha, B) \in D_1$, D_2, and D_4. When $(\alpha, B) \in D_1$, Lemma 2.10 and Theorem 2.11 show that each of g' and g has at most one positive zero. When $(\alpha, B) \in D_4$, Theorem 2.4 shows that each of g and g' has exactly one positive zero. The numerical computation shows that the corresponding TPBVP possesses 2-cell solutions when $(\alpha, B) \in D_1, D_4$. When $(\alpha, B) \in D_2$, Theorem 2.12 shows that g has at most two positive zeros. The numerical computation shows that g has two zeros for $B \in (0.381, 1)$. Let y_1 and y_2 be the zeros. The function $g(y)$, where $y \in [0, y_1]$, corresponds to a two-cell solution, and $g(y)$, where $y \in [0, y_2]$, corresponds to a three-cell solution.

Next, we let $A = 4$, and Fig. 1(b) shows the bifurcation diagram for $(Q, \beta) \in [-6000, 10000] \times [0, 180]$. From Theorems 2.2 and 2.7, g has no zero when $(\alpha, B) \in D_1$ and D_3. Lemma 2.8 and Theorems 2.5 and 2.9 show that both g and g' have at most one zero when (α, B) in D_2 and D_4. The zeros of g are computed for $(\alpha, B) \in D_2$ and D_4. The function g has one zero for $B \in (-1, -0.348)$ and $B \in (0.348, 1)$, and the corresponding TPBVP possesses 2-cell solutions.

4 CONCLUSION

In this paper, the existence of solutions for the TPBVP, given by Eqs. (1.1) and (1.2), is studied. The TPBVP is first transformed into an IVP which is presented by Eqs. (2.3) and (2.4). Solving the zeros of the solution g to Eqs. (2.3) and (2.4) is equivalent to solving the TPBVP. In this paper, the existence properties of the zeros of g have been proven for $A \geq 1$. From the mathematical analysis, we conclude that the TPBVP possesses only 2-cell solutions when $A \geq 2$, and it may possess 2-cell and 3-cell solutions when $1 \leq A < 2$. A homogeneity property of parameters is proven so that the numerical computation on the parameter $\alpha - B$ plane is reduced to the perimeter of the square $|\alpha| + |B| = 1$. This greatly improves the computational efficiency. Numerical simulation is then conducted to verify the existence property of solutions for the TPBVP.

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COMPETING INTERESTS

The authors declare that no competing interests exist.

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