Asian Research Journal of Mathematics

Volume 18, Issue 12, Page 120-127, 2022; Article no.ARJOM.95042
ISSN: 2456-477X

Analysis of Hybrid Stochastic Gompertz Model with Time Delay

Guixin Hu [a] and Bingqing Li [a*]

[a] *School of Mathematics and Information Science, Henan Polytechnic University (HPU), Jiaozuo 454000, P. R. China.*

Authors' contributions

This work was carried out in collaboration between both authors. Author GH designed the study, wrote the protocol, and wrote the first draft of the manuscript. Author BL was responsible for the analysis of the study. Authors GH and BL jointly managed the literature search. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ARJOM/2022/v18i12631

Open Peer Review History:
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://www.sdiarticle5.com/review-history/95042

Received: 20/10/2022
Accepted: 23/12/2022
Published: 28/12/2022

Original Research Article

Abstract

This study explores a hybrid stochastic delay Gompertz model under regime switching. It is proved that the model has a unique global positive solution. Sufficient conditions for persistence in mean and extinction are obtained. The results show that the random perturbations and time delays could effect the persistence and extinction of the model.

Keywords: Gompertz equation; random perturbations; persistence in mean; extinction; Markov chains.

2010 Mathematics Subject Classification: 60H10, 92B05, 81T80.

**Corresponding author: E-mail: lbqing1999@163.com;*

1 Introduction

1.1 Background and research aims

Tumors have always been a major threat to our human health. More than 9.5 million people were counted to have died from the tumors in 2018 [1]. The growth of the tumor cells could be modeled by the Gompertz equation. Let $\psi(t)$ be the number of cells, it has the following form

$$\mathrm{d}\psi(\iota) = (a\psi(\iota) - b\psi(\iota)\ln\psi(\iota))\mathrm{d}\iota, \qquad \iota > 0, \tag{1.1}$$

here $a,\ b$ denotes the intrinsic growth rate and growth deceleration factor of the tumor, respectively.

To be better model the process, one can consider the time delays which represent the lag in the tumor growth/regression process. In [2], the authors studied the previous time $\iota - \tau$ to determine the per capita growth at the current time t, i.e.

$$\mathrm{d}\psi(\iota) = [a\psi(\iota) - b\psi(\iota)\ln\psi(\iota - \tau)]\mathrm{d}\iota, \qquad \iota > 0, \tag{1.2}$$

with the initial date $\psi_0 = \{\varrho(\varsigma), -\tau \le \varsigma \le 0\}$ where $\varrho(\iota)$ is continuous function from $[-\tau, 0]$ to R^+.

Stochastic perturbations are common in our daily life [3, 4, 5]. They can usually be divided into large and small perturbations, May RM [6] stated that the disturbances in the environment have great impacts on the growth of the species, it can be estimated by modeling methods. This methodology has been widespread adoption (see [7, 8, 9, 10, 11, 12, 13]). Following this approach, $a \to a + \sigma\dot{B}(\iota)$, where $B(\iota)$ denotes the standard Brownian motion and the constant σ represents the strength of the white noise, one can get the following stochastic Gompertz model with time delay as

$$\mathrm{d}\psi(\iota) = [a\psi(\iota) - b\psi(\iota)\ln\psi(\iota - \tau)]\mathrm{d}\iota + \sigma\psi(\iota)dB(\iota), \qquad t > 0. \tag{1.3}$$

In addition to the perturbations described above, there are other perturbations (e.g., drug concentration, oxygen supply) that can cause species to change their state, such as their growth switching from one state to another, however, this variation must not be estimated with the white noise [7]. For example, the mortality rate of recently hatched small yellow croaker varies at different temperatures [14]. In general, the next state switching is not the same as the one before, and the time at which the switching occurs follows an exponential distribution [8, 9, 10]. Therefore, the Markov chains $\varpi(\iota)$ can be used to model this regime switchings [8, 9, 10].

In this paper, we mainly consider $a = r,\ b = r\beta$ in model (1.3). Thus, with model (1.3), we can get the hybrid stochastic system as

$$\mathrm{d}\psi(\iota) = r(\varpi(\iota))[\psi(\iota) - \beta(\varpi(\iota))\psi(\iota)\ln\psi(\iota - \tau)]\mathrm{d}\iota + \sigma(\varpi(\iota))\psi(\iota)\mathrm{d}B(\iota), \qquad \iota > 0, \tag{1.4}$$

where $r_i > 0,\ \beta_i > 0,\ \sigma_i > 0$ for any $i \in \mathbb{S}$, the Markov chain $\varpi(\iota)$ is independent of $B(\iota)$.

Some integral differential equations are difficult to solve exactly, and in [15, 16] A.Hamoud et al. studied the behavior of the solution by analyzing the approximation form, and in [17, 18, 19, 20] the convergence and uniqueness of the solution were analyzed. Of course, the solution of stochastic differential equations is also difficult to obtain. In this paper, we get the existence of the solution to model (1.4). We will focus on the asymptotic behaviors of the model.

This paper consists of the following structures: first, we show the existence and positivity of the solution for Eq. (1.4). Secondly, in Section 3 we give the persistence in mean and extinction of Eq. (1.4) with the time delays. Finally, we concluded the paper with an example and a brief discussion.

2 The Existence of Positive Solutions

Through out this paper, we always assume that the Markov chain $\varpi(\iota)$ is irreducible. It means that the following linear equation (see, [12, 13])

$$\Upsilon Q = 0, \qquad \sum_{i=1}^{N} \Upsilon_i = 1, \tag{2.1}$$

has a single fixed solution $\Upsilon = (\Upsilon_1, \cdots, \Upsilon_N)$ which satisfys $\Upsilon_i > 0,\ i \in \mathbb{S}$.

To proceed with our discussion, we need the following notations:

$$\acute{r} = \max_{1\leq i\leq N} r_i,\ \grave{r} = \min_{1\leq i\leq N} r_i,\ \acute{\beta} = \max_{1\leq i\leq N} \beta_i,$$
$$\grave{\beta} = \min_{1\leq i\leq N} \beta_i,\ \omega_i = r_i - \frac{1}{2}\sigma_i^2.$$

Theorem 2.1. *Suppose that Eq. (1.4) satisfies the following condition*

$$\max_{i\in\mathbb{S}} |\omega_i + r_i\beta_i| \leq M. \tag{2.2}$$

Then Eq. (1.4) has a unique positive solution $\psi(\iota)$.

Proof. Consider the following differential equation

$$\mathrm{d}z(\iota) = [\omega(\varpi(\iota)) - r(\varpi(\iota))\beta(\varpi(\iota))z(\iota-\tau)]\mathrm{d}\iota + \sigma(\varpi(\iota))x(\iota)\mathrm{d}B(\iota), \tag{2.3}$$

with the initial value $z_0 = \ln\psi_0$. It is easy to see that under the certain assumptions, Eq. (2.3) satisfies the global Lipschitz condition and the linear growth condition. Next, define $\psi(\iota) = e^{z(\iota)}$ and using Itô formula, one can obtain that

$$\begin{aligned}\mathrm{d}\psi(\iota) &= e^{z(\iota)}[\omega(\varpi(\iota)) - r(\varpi(\iota))\beta(\varpi(\iota))z(\iota-\tau)]\mathrm{d}(\iota) + e^{z(\iota)}\sigma(\varpi(\iota))\mathrm{d}B(\iota)\\ &\quad + \frac{1}{2}e^{z(\iota)}\sigma^2(\varpi(\iota))\mathrm{d}\iota\\ &= \psi(\iota)[r(\varpi(\iota)) - r(\varpi(\iota))\beta(\varpi(\iota))\ln\psi(\iota-\tau)]\mathrm{d}\iota + \psi(\iota)\sigma(\varpi(\iota))\mathrm{d}B(\iota).\end{aligned} \tag{2.4}$$

The proof is compete. □

3 Persistence in Mean and Extinction

In this section, we are going to study the survival and extinction of Eq. (1.4). The definitions for persistence in mean and extinction for stochastic model could be found in [21, 22].

Definition 3.1. Suppose that $\psi(\iota)$ is a solution of Eq. (1.4), then
(i) $\psi(\iota)$ is **persistence in mean** if $\liminf_{\iota\to\infty} \frac{1}{\iota}\int_0^{\iota} \psi(s)\mathrm{d}s > 0$ *a.s.*;
(ii) $\psi(\iota)$ is **extinction** if $\lim_{\iota\to\infty} \psi(\iota) = 0$ *a.s.*.

Theorem 3.1. *Suppose Theorem 2.1 holds and*

$$h_* = \sum_{i=1}^{N} \pi_i[\omega_i + r_i\beta_i] > 0. \tag{3.1}$$

Then Eq. (1.4) is persistence in mean.

Proof. By using the generalised Itô formula to Eq. (1.4), we can get that

$$\ln\psi(\iota) = \ln\psi(0) + \int_0^\iota [\omega(\varpi(\hbar)) - r(\varpi(\hbar))\beta(\varpi(\hbar))\ln\psi(\hbar-\tau)]\mathrm{d}\hbar + \int_0^\iota \sigma(\varpi(\hbar))\mathrm{d}B(\hbar). \tag{3.2}$$

Elementary inequality $\ln\psi \le \psi - 1$ for $\psi > 0$, implies that

$$\begin{aligned}\ln\psi(\iota) \quad &+ \int_0^\iota r(\varpi(\hbar))\beta(\varpi(\hbar))\ln\psi(\hbar-\tau)\mathrm{d}\hbar \\ &\le \psi(\iota) + \int_0^\iota r(\varpi(\hbar))\beta(\varpi(\hbar))\psi(\hbar-\tau)\mathrm{d}\hbar - \int_0^\iota r(\varpi(\hbar))\beta(\varpi(\hbar))\mathrm{d}\hbar \\ &\le \psi(\iota) + \acute{r}\acute{\beta}\int_0^\iota \psi(\hbar)\mathrm{d}\hbar - \int_0^\iota r(\varpi(\hbar))\beta(\varpi(\hbar))\mathrm{d}\hbar + \acute{r}\acute{\beta}\int_{-\tau}^0 \varrho(\varpi(\hbar))\mathrm{d}\hbar \\ &= e^{-\acute{r}\acute{\beta}\iota}\frac{d}{d\iota}\Big(e^{\acute{r}\acute{\beta}\iota}\int_0^\iota \psi(\hbar)\mathrm{d}\hbar\Big) - \int_0^\iota r(\varpi(\hbar))\beta(\varpi(\hbar))\mathrm{d}\hbar + \acute{r}\acute{\beta}\int_{-\tau}^0 \varrho(\varpi(\hbar))\mathrm{d}\hbar.\end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3), we obtain that

$$\begin{aligned}\ln\psi(0) \quad &+ \int_0^\iota \omega(\varpi(\hbar))\mathrm{d}\hbar + \int_0^\iota \sigma(\varpi(\hbar))\mathrm{d}B(\hbar) \\ &\le \psi(\iota) + \acute{r}\acute{\beta}\int_0^\iota \psi(\hbar)\mathrm{d}\hbar - \int_0^\iota r(\varpi(\hbar))\beta(\varpi(\hbar))\mathrm{d}\hbar + \acute{r}\acute{\beta}\int_{-\tau}^0 \varrho(\varpi(\hbar))\mathrm{d}\hbar.\end{aligned} \tag{3.4}$$

Therefore,

$$\begin{aligned}e^{-\acute{r}\acute{\beta}\iota}\frac{\mathrm{d}}{\mathrm{d}\iota}\Big(e^{\acute{r}\acute{\beta}\iota}\int_0^\iota \psi(\hbar)\mathrm{d}\hbar\Big) \quad &\ge \ln\psi(0) + \int_0^\iota [\omega(\varpi(\hbar)) + r(\varpi(\hbar))\beta(\varpi(\hbar))]\mathrm{d}\hbar \\ &+ \int_0^\iota \sigma(\varpi(\hbar))\mathrm{d}B(\hbar) - \acute{r}\acute{\beta}\int_{-\tau}^0 \varrho(\varpi(\hbar))\mathrm{d}\hbar.\end{aligned} \tag{3.5}$$

Now, integrating both sides of (3.5), it yields that

$$\begin{aligned}\int_0^\iota \psi(\hbar)\mathrm{d}\hbar \quad &\ge \frac{C}{\acute{r}\acute{\beta}}(1 - e^{-\acute{r}\acute{\beta}\iota}) + \int_0^\iota \Big(e^{\acute{r}\acute{\beta}(\hbar-\iota)}\int_0^\hbar [\omega(\varpi(u)) + r(\varpi(u))\beta(\varpi(u))]\mathrm{d}u\Big)\mathrm{d}\hbar \\ &+ \int_0^\iota \Big(e^{\acute{r}\acute{\beta}(\hbar-\iota)}\int_0^\hbar \sigma(\varpi(u))\mathrm{d}B(u)\Big)d\hbar \\ &= \frac{C}{\acute{r}\acute{\beta}}\big(1 - e^{-\acute{r}\acute{\beta}\iota}\big) + \frac{1}{\acute{r}\acute{\beta}}\int_0^\iota [\omega(\varpi(\hbar)) + r(\varpi(\hbar))\beta(\varpi(\hbar))]\mathrm{d}\hbar \\ &- \frac{1}{\acute{r}\acute{\beta}}\int_0^\iota e^{\acute{r}\acute{\beta}(\hbar-\iota)}[\omega(\varpi(\hbar)) + r(\varpi(\hbar))\beta(\varpi(\hbar))]\mathrm{d}\hbar \\ &+ \frac{1}{\acute{r}\acute{\beta}}\int_0^\iota \sigma(\varpi(\hbar))\mathrm{d}B(\hbar) - \frac{1}{\acute{r}\acute{\beta}}\int_0^\iota e^{\acute{r}\acute{\beta}(\hbar-\iota)}\sigma(\varpi(\hbar))\mathrm{d}B(\hbar),\end{aligned} \tag{3.6}$$

where $C = \ln\psi(0) - \acute{r}\acute{\beta}\int_{-\tau}^0 \varrho(\varpi(\hbar))\mathrm{d}\hbar$. On the other hand, let

$$M_1(\iota) = \int_0^\iota \sigma(\varpi(\hbar))\mathrm{d}B(\hbar), M_2(\iota) = \int_0^\iota e^{\acute{r}\acute{\beta}(\hbar-\iota)}\sigma(\varpi(\hbar))\mathrm{d}B(\hbar). \tag{3.7}$$

Note that $M(\iota)$ is a martingale with quadratic variation

$$\langle M_1(\iota), M_1(\iota)\rangle = \int_0^\iota \sigma^2(\varpi(\hbar))\mathrm{d}\hbar \le \acute{\sigma}\iota, \tag{3.8}$$

$$\langle M_2(\iota), M_2(\iota)\rangle = \int_0^\iota e^{2\acute{r}\acute{\beta}(\hbar-\iota)}\sigma^2(\varpi(\hbar))\mathrm{d}\hbar \le \acute{\sigma}\iota. \tag{3.9}$$

Using the strong law of large numbers for local martingales (see, e.g., [23]), we have

$$\lim_{\iota\to\infty}\frac{M_i(\iota)}{\iota} = 0 \qquad a.s.,\ i = 1, 2. \tag{3.10}$$

From this we see that

$$\begin{aligned}\liminf_{\iota\to\infty}\frac{1}{\iota}\int_0^{\iota}\psi(\hbar)\mathrm{d}\hbar &\geq \liminf_{\iota\to\infty}\frac{1}{\iota\acute{r\beta}}\int_0^{\iota}[\omega(\varpi(\hbar))+r(\varpi(\hbar))\beta(\varpi(\hbar))]\mathrm{d}\hbar\\ &-\liminf_{\iota\to\infty}\frac{1}{\iota\acute{r\beta}}\int_0^{\iota}e^{\acute{r\beta}(\hbar-\iota)}[\omega(\varpi(\hbar))+r(\varpi(\hbar))\beta(\varpi(\hbar))]\mathrm{d}\hbar.\end{aligned} \tag{3.11}$$

Since

$$\lim_{\iota\to\infty}\frac{\left|\int_0^{\iota}e^{\acute{r\beta}(\hbar-\iota)}[\omega(\varpi(\hbar))+r(\varpi(\hbar))\beta(\varpi(\hbar))]\mathrm{d}\hbar\right|}{\iota}\leq\lim_{\iota\to\infty}\frac{M(1-e^{-\acute{r\beta}\iota})}{\acute{r\beta}\iota}=0, \tag{3.12}$$

holds a.s.. Then from (3.11)

$$\liminf_{\iota\to\infty}\frac{1}{\iota}\int_0^{\iota}\psi(\hbar)\mathrm{d}\hbar\geq\liminf_{\iota\to\infty}\frac{1}{\iota}\int_0^{\iota}[\omega(\varpi(\hbar))+r(\varpi(\hbar))\beta(\varpi(\hbar))]\mathrm{d}\hbar=\sum_{i=1}^{N}\pi_i[\omega_i+r_i\beta_i]=h_*. \tag{3.13}$$

Now, if $h_*>0$, we have

$$\liminf_{\iota\to\infty}\frac{1}{\iota}\int_0^{t}\psi(\hbar)\mathrm{d}\hbar>0\quad a.s.. \tag{3.14}$$

This completes the proof.

□

Theorem 3.2. *Suppose*

$$\max_{i\in\mathbb{S}}|r_i-\frac{\sigma_i^2}{2}|\leq A, \tag{3.15}$$

and

$$\eta=\lim_{\iota\to\infty}\int_0^{\iota}r(\varpi(u))\beta(\varpi(u))\mathrm{d}u<1, \tag{3.16}$$

hold, and for all $\iota\geq 0$

$$r-\frac{\sigma^2}{2}\leq-\theta<0, \tag{3.17}$$

where θ *is a constant such that* $\theta>\frac{\eta A}{1-\eta}$*. Then Eq. (1.4) is extinction with probability* 1.

Proof. By using Itô formula to Eq. (1.4), we show that

$$\ln\psi(\iota)=\ln\psi(0)+\int_0^{\iota}[r(\varpi(\hbar))-\frac{1}{2}\sigma^2(\varpi(\hbar))-\beta(\varpi(\hbar))r(\varpi(\hbar))\ln\psi(\hbar-\tau)]\mathrm{d}\hbar+\int_0^{\iota}\sigma(\varpi(\hbar))\mathrm{d}B(\hbar). \tag{3.18}$$

Consequently,

$$\begin{aligned}|\ln\psi(\iota)| &\leq|\ln\psi(0)|+\int_0^{\iota}|r(\varpi(\hbar))-\frac{1}{2}\sigma^2(\varpi(\hbar))|\mathrm{d}\hbar\\ &+\int_0^{\iota}\beta(\varpi(\hbar))r(\varpi(\hbar))|\ln\psi(\hbar-\tau)|\mathrm{d}\hbar+|\int_0^{\iota}\sigma(\varpi(\hbar))\mathrm{d}B(\hbar)|\\ &\leq|\ln\psi(0)|+A\iota+\eta\sup_{u\in[-\tau,\iota]}\{|\ln\psi(u)|\}+|\int_0^{\iota}\sigma(\varpi(\hbar))\mathrm{d}B(\hbar)|.\end{aligned} \tag{3.19}$$

It follows from (3.19) that

$$\begin{aligned}\sup_{u\in[-\tau,\iota]}\{|\ln\psi(u)|\} &\leq\sup_{u\in[-\tau,0]}\{|\ln\psi(u)|\}+\sup_{u\in[0,\iota]}\{|\ln\psi(u)|\}\\ &\leq 2\sup_{u\in[-\tau,0]}\{|\ln\psi(u)|\}+A\iota+\eta\sup_{u\in[-\tau,\iota]}\{|\ln\psi(u)|\}\\ &+\sup_{u\in[0,\iota]}|\int_0^{u}\sigma(\varpi(\hbar))\mathrm{d}B(\hbar)|,\end{aligned} \tag{3.20}$$

and then

$$\begin{aligned}\sup_{u\in[-\tau,\iota]}\{|\ln\psi(u)|\} \leq & \frac{2}{1-\eta}\sup_{u\in[-\tau,0]}\{|\ln\psi(u)|\} \\ & +\frac{A}{1-\eta}\iota+\frac{1}{1-\eta}\sup_{u\in[0,\iota]}\left|\int_0^u\sigma(\varpi(\hbar))\mathrm{d}B(\hbar)\right|.\end{aligned}\tag{3.21}$$

This, together with (3.18), gives that

$$\begin{aligned}\ln\psi(\iota) & \leq \ln\psi(0)+(-\theta\iota)+\int_0^\iota\sigma(\varpi(\hbar))\mathrm{d}B(\hbar)+\frac{2\eta}{1-\eta}\sup_{u\in[-\tau,0]}\{|\ln\psi(u)|\}+\frac{A\eta}{1-\eta}\iota \\ & +\frac{\eta}{1-\eta}\sup_{u\in[0,\iota]}\left|\int_0^u\sigma(\xi(\hbar))\mathrm{d}B(\hbar)\right| \\ & \leq \frac{1+\eta}{1-\eta}\sup_{u\in[-\tau,0]}\{|\ln\psi(u)|\}+(-\theta+\frac{A\eta}{1-\eta})+\frac{1}{1-\eta}\sup_{u\in[0,\iota]}\left|\int_0^u\sigma(\varpi(\hbar))\mathrm{d}B(\hbar)\right|.\end{aligned}\tag{3.22}$$

Then by using the strong law of large numbers for martingales, from (3.22) we obtain that

$$\limsup_{\iota\to\infty}\frac{\ln\psi(\iota)}{\iota}\leq-\theta+\frac{\eta A}{1-\eta}<0.\tag{3.23}$$

That is

$$\lim_{\iota\to\infty}\psi(\iota)=0\quad a.s..\tag{3.24}$$

The proof is complete. □

4 Example and Conclusion

Example: Suppose that the irreducible Markov chain $\varpi(\iota)$ taking values in $\mathbb{S}=\{1,2\}$ and the transition probability matrix be $q=\begin{pmatrix}-1 & 1\\ 2 & -2\end{pmatrix}$. Thus, its stationary distribution is $\Upsilon=(\Upsilon_1,\Upsilon_2)=(\frac{2}{3},\frac{1}{3})$. To verify the result of Theorem 3.1, let

$$\begin{aligned}& r_1=4,\ r_2=1,\ \sigma_1=\sqrt{5},\ \sigma_2=\sqrt{3},\\ & \beta_1=1/2,\ \beta_2=1,\ \omega_1=1.5,\ \omega_2=-0.5,\ M=3.5892.\end{aligned}$$

By calculation, we can get

$$h_*=\pi_1[\omega_1+r_1\beta_1]+\pi_2[\omega_2+r_2\beta_2]>0,\ \omega_2+r_2\beta_2\leq M.$$

Therefore, by Theorem 3.1, Eq. (1.4) is persistence in mean.

An important topic in ecology is the effect of various perturbations on the persistence in mean and extinction of stochastic Gompertz models [6]. In this paper we present and explore the stochastic Gompertz model with two kinds of stochastic perturbations and time delays. Theorem 3.1 and Theorem 3.2 establish the sufficient conditions for persistence in mean and extinction of the species.

The topics that trigger our further research through this article are as follows: firstly what happens if Eq. (1.4) is perturbed by the intrinsic growth rate r while β is also perturbed by white noise. Secondly what happens to the asymptotic properties of Eq. (1.4) if both distribution delay and Lévy jumps are introduced (for more details of Lévy jumps, see [11, 24, 25]).

Funding

This research work was supported by the Foundation of Henan Educational Committee (23A110009) and the Program of Young Scholar for Henan Polytechnic University (2020XQG-03).

Acknowledgement

The authors extend their appreciations to the anonymous referees for their very helpful suggestions which greatly improve the paper.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Bray F, Ferlay J, Soerjomataram I, et al. Global cancer statistics 2018: GLOBOCAN estimates of incidence and mortality worldwide for 36 cancers in 185 countries. CA: Cancer J Clin. 2018;68;394-424. DOI:10.3322/caac.21492.

[2] Piotrowska MJ, Forys U. The nature of Hopf bifurcation for the Gompertz model with delays. Math Comput Model. 2011;54(9-10):2183-2198. DOI:10.1016/j.mcm.2011.05.027.

[3] Krstic M, Jovanovic M. On stochastic population model with the Allee effect. Math Comput Modelling. 2010;52(1-2):370-379. DOI:10.1016/j.mcm.2010.02.051.

[4] Carlos C, Braumann CA. General population growth models with Allee effects in a random environment. Ecol Complex. 2017;30:26-33. DOI:10.1016/j.ecocom.2016.09.003.

[5] Zhang B, Wang H, Lv G. Exponential extinction of a stochastic predator-prey model with Allee effect. Physica A. 2018;507:192-204. DOI:10.1016/j.physa.2018.05.073.

[6] May RM. Stability and Complexity in Model Ecosystems. Princeton Univ Press; 1973.

[7] Wang K. Stochastic Mathematical Biology Models. Science Press Beijing; 2010.

[8] Li D, Liu M. Invariant measure of a stochastic food-limited population model with regime switching. Math Comput Simul. 2020;178:16-26. DOI:10.1016/j.matcom.2020.06.003.

[9] Liu M. Dynamics of a stochastic regime-switching predator-prey model with modified Leslie-Gower Holling-type II schemes and prey harvesting. Nonlinear Dynam. 2019;96:417-442. DOI:10.1007/s11071-019-04797-x.

[10] Yu X, Yuan S, Zhang T. Persistence and ergodicity of a stochastic single species model with Allee effect under regime switching. Commun Nonlinear Sci Numer Simul. 2018;59:359-374. DOI:10.1016/j.cnsns.2017.11.028.

[11] Xu J, Wang Y, Cao Z. Dynamics of a stochastic SIRS epidemic model with standard incidence under regime switching. Int J Biomath. 2022;15(02):2150074.

[12] Liu M, Deng M. Permanence and extinction of a stochastic hybrid model for tumor growth. Appl Math Lett. 2019;94:66-72. DOI:10.1016/j.aml.2019.02.016.

[13] Liu M, Bai C. Optimal harvesting of a stochastic mutualism model with regime-switching. Appl Math Comput. 2020;375:125040. DOI:10.1016/j.amc.2020.125040.

[14] Sun Y, Xie Q, Lou B, He X, et al. Effect of temperature on early growth stage in Little Yellow Croaker. Larimichthys polyactis, J Zhejiang Ocean Univ. 2018;37:208-214.

[15] Issa MB, Hamoud A, Sharif A, Ghadle K, Giniswamy G. Modified Adomian decomposition method for solving fuzzy integro-differential equations. Canad J Appl Math. 2021;3(1):37-45.

[16] Hamoud A, Azeez A, Ghadle K. A study of some iterative methods for solving fuzzy Volterra-Fredholm integral equations. Indonesian Journal of Electrical Engineering and Computer Science. 2018;11(3):1228-1235.

[17] Hamoud A, Ghadle K. Modified Laplace decomposition method for fractional Volterra-Fredholm integro-differential equations. Journal of Mathematical Modeling. 2018;6(1):91-104.

[18] Dawood L, Sharif A, Hamoud A. Solving higher-order integro differential equations by VIM AND MHPM. International Journal of Applied Mathematics. 2020;33(2):253-264.

[19] Hamoud A, Ghadle K. Usage of the variational iteration technique for solving Fredholm integro-differential equations. Journal of Computational Applied Mechanics. 2019;50(2):303-307.

[20] Hamoud A, Dawood A, Ghadle K, Atshan S. Usage of the modified variational iteration technique for solving Fredholm integro-differential equations. International Journal of Mechanical and Production Engineering Research and Development 2019;9(2):895-902.

[21] Ji C, Jiang D, Li X. Qualitative analysis of a stochastic ratio-dependent predator-prey system. J Comput Appl Math. 2011;235(5):1326-1341.
DOI:10.1016/j.cam.2010.08.021

[22] Liu M, Wang K. Persistence and extinction in stochastic non-autonomous logistic systems. J Math Anal Appl. 2011;375(2):443-457.
DOI:10.1016/j.jmaa.2010.09.058.

[23] Mao XY, Yuan CG. Stochastic Differential Equations with Markovian Switching. Imperial College Press; 2006.

[24] Bao J, Mao X, Yin G, Yuan C. Competitive Lotka-Volterra population dynamics with jumps. Nonlinear Anal. 2011;74(17):6601-6616.
DOI:10.1016/j.na.2011.06.043.

[25] Zhang X, Chen F, Wang K, Du H. Stochastic SIRS model driven by Lévy noise. Acta Math Sci. 2016;36(3):740-752.
DOI:10.1016/s0252-9602(16)30036-4.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
https://www.sdiarticle5.com/review-history/95042