Second Order Rotatable Designs of Second Type Using Pairwise Balanced Designs

P. Chiranjeevi[1*] and B. Re. Victorbabu[1]

[1]*Department of Statistics, Acharya Nagarjuna University, Guntur-522510, A. P., India.*

Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

Original Research Article

Abstract

In this paper, second order rotatable designs of second type using pairwise balanced designs is suggested. This design is compared with second order rotatable designs of first type using pairwise balanced designs [1] on the basis of efficiency.

1 Introduction

Response surface designs is a collection of mathematical and statistical techniques useful for analyzing problems where several independent variables influence a dependent variable or response. Box and Hunter [2] introduced designs having spherical variance function are called rotatable designs. Das and Narasimham [3] constructed rotatable designs using balanced incomplete block designs (BIBD). Raghavarao [4] constructed second order rotatable designs (SORD) using incomplete block designs. Draper and Guttman [5] suggested an index of rotatability. Khuri [6] introduced a measure of rotatability for response surface designs. Draper and Pukelshein [7] developed another look at rotatability. Park et al. [8] introduced new

---

*Corresponding author: E-mail: chirupunugupati@gmail.com;

measure of rotatability for second order response surface designs. Das et al. [9] developed modified response surface designs. Kim [10] introduced extended central composite designs (CCD) with the axial points are indicated by two numbers. Victorbabu and Vasundharadevi [11] suggested modified second order response surface designs using BIBD. Victorbabu [12] constructed modified SORD using a pair of BIBD. Victorbabu et al. [13] studied modified second order response surface designs using pairwise balanced designs (PBD). Victorbabu [14] suggested a review on second order rotatable designs. Victorbabu et al. [14] suggested modified second order response surface designs using CCD. Victorbabu and Vasundharadevi [14] studied second order response surface designs using symmetrical unequal block arrangements (SUBA) with two unequal block sizes. Victorbabu [15] constructed modified SORD using a pair of SUBA with two unequal block sizes. Park and Park [16] suggested the extension of CCD for second order response surface models. Victorbabu and Surekha [17] suggested measure of rotatability for second order response surface designs using incomplete block designs. Victorbabu and Surekha [18] developed measure of rotatability for second order response surface designs using BIBD. Victorbabu et al. [19,20] studied measure of rotatability for second order response surface designs using a pair SUBA with two unequal block sizes and pair of BIBD. Kim [21] suggested modified slope rotatability using extended CCD. Jyostna and Victorbabu [22,23,24] studied measure of modified rotatability for second degree polynomials using BIBD, PBD and SUBA with two unequal block sizes. Jyostna et al. [25] suggested measure of modified rotatability for second order response surface designs using CCD. Chiranjeevi et al. [26] extended the work of Kim [10] and suggested second order rotatable designs of second type using CCD for $9 \leq v \leq 17$ (v: number of factors). Chiranjeevi and Victorbabu [27,28] developed SORD second type using SUBA with two unequal block sizes and BIBD.

In this paper, second order rotatable designs of second type using pairwise balanced designs is suggested. This design is compared with second order rotatable designs of first type using pairwise balanced designs [1] on the basis of efficiency.

2 Stipulations and Formulas for Second Order Rotatable Designs

Suppose we want to use the second order polynomial response surface design $D = ((x_{iu}))$ to fit the surface,

$$Y_u = b_0 + \sum_{i=1}^{v} b_i x_{iu} + \sum_{i=1}^{v} b_{ii} x_{iu}^2 + \sum_{i<j} \sum b_{ij} x_{iu} x_{ju} + \varphi_u \quad (2.1)$$

where x_{iu} represents the level of ith factor (i=1,2,…,v) in the uth run (u=1,2,…,N) of the experiment and φ_u are uncorrelated random error with mean zero and variance σ^2. Then 'D' is said to be second order rotatable designs (SORD) if the variance of $Y_u(x_1, x_2, …, x_v)$ with respect to each of independent variable (x_i) is only a function of the distance $(d^2 = \sum_{i=1}^{v} x_i^2)$ of the point $(x_1, x_2, …, x_v)$ from the origin (center) of the design. Such a spherical variance function for estimation of responses in the second order polynomial model is achieved if the design points satisfy the following conditions [29].

All odd order moments are zero. In their words when at least one odd power x's equal to zero.

1. $\sum x_{iu} = 0, \sum x_{iu} x_{ju} = 0, \sum x_{iu} x_{ju}^2 = 0, \sum x_{iu} x_{ju} x_{ku} = 0,$
   $\sum x_{iu}^3 = 0, \sum x_{iu} x_{ju}^3 = 0, \sum x_{iu} x_{ju} x_{ku}^2 = 0, \sum x_{iu} x_{ju} x_{ku} x_{lu} = 0.$
   for $i \neq j \neq k \neq l$; $\quad (2.2)$

2. (i) $\sum x_{iu}^2 = \text{constant} = N\mu_2$
   (ii) $\sum x_{iu}^4 = \text{constant} = cN\mu_4$ for all i $\quad (2.3)$

$$3. \sum x_{iu}^2 x_{ju}^2 = \text{constant} = N\mu_4; \text{ for all } i \neq j \tag{2.4}$$

$$4. \frac{\mu_4}{\mu_2^2} > \frac{v}{(c+v-1)} \tag{2.5}$$

$$5. \sum x_{iu}^4 = c \sum x_{iu}^2 x_{ju}^2 \tag{2.6}$$

where c, μ_4 and μ_2 are constants.

The variances and covariances of the estimated parameters are

$$V(\hat{b}_0) = \frac{\mu_4(c+v-1)\sigma^2}{N[\mu_4(c+v-1)-v\mu_2^2]},$$

$$V(\hat{b}_i) = \frac{\sigma^2}{N\mu_2},$$

$$V(\hat{b}_{ij}) = \frac{\sigma^2}{N\mu_4},$$

$$V(\hat{b}_{ii}) = \frac{\sigma^2}{(c-1)N\mu_4} \left[\frac{\mu_4(c+v-2)-(v-1)\mu_2^2}{\mu_4(c+v-1)-v\mu_2^2} \right],$$

$$Cov(\hat{b}_0, \hat{b}_{ii}) = \frac{-\mu_2\sigma^2}{N[\mu_4(c+v-1)-v\mu_2^2]},$$

$$Cov(\hat{b}_{ii}, \hat{b}_{jj}) = \frac{(\mu_2^2-\mu_4)\sigma^2}{(c-1)N\mu_4[\mu_4(c+v-1)-v\mu_2^2]} \quad \text{and other covariances vanish.} \tag{2.7}$$

The variance of the estimated response at the point $(x_{10}, x_{20}, \ldots, x_{v0})$ is

$$V(\hat{y}_0) = V(\hat{b}_0) + \left[V(\hat{b}_i) + 2Cov(\hat{b}_0, \hat{b}_{ii}) \right] d^2 + V(\hat{b}_{ii})d^4 +$$
$$\sum x_{i0}^2 x_{j0}^2 \left[V(\hat{b}_{ij}) + 2Cov(\hat{b}_{ii}, \hat{b}_{jj}) - 2V(\hat{b}_{ii}) \right] \tag{2.8}$$

The coefficient of $\sum x_{i0}^2 x_{j0}^2$ in the above equation (2.8) is simplified to $(c-3)\sigma^2/(c-1)N\mu_4$.

A second order response surface design D is said to be SORD, if in this design c=3 and all the other conditions (2.2) to (2.7) hold.

3 SORD of First Type Using Pairwise Balanced Designs (cf. Tyagi [1]).

Let $(v, b, r, k_1, k_2, .., k_p, \lambda)$ denote parameters of PBD, $k=\sup(k_1,k_2,..,k_p)$ and $2^{t(k)}$ denote a fractional replication of 2^k in +1 or -1 levels in which no interaction with less than five factors is confounded. $[1-(v, b, r, k_1, k_2, \ldots, k_p, \lambda)]$ denote the design points generated from transpose of the incidence matrix of PBD. Let $[1-(v,b,r,k_1,k_2,...,k_p,\lambda)]2^{t(k)}$ are the $b2^{t(k)}$ design points generated from PBD by "multiplication" (cf. Raghavarao [30], pp 298-300), $(\pm a, 0, 0, \ldots, 0)2^1$ denote the design points generated from $(\pm a, 0, 0, \ldots, 0)$ point set. Let U denote the union of the design points generated from different sets of points, n_0 denote the number of central points. The method of construction of SORD of first type using PBD is given in the following result.

Result: The design points, $[1-(v,b,r,k_1,k_2,...,k_p,\lambda)]2^{t(k)} U (a,0,...,0)2^1 U (n_0)$ will give a v-dimensional SORD of first type using PBD in $N=b2^{t(k)}+2v+n_0$ design points, with

$$a^4 = \frac{2^{t(k)}(3\lambda-r)}{2}.$$

The condition for the design will becomes an orthogonal design.

From equation 2 (i) of (2.3) and (3) of (2.4), we have

$$\sum x_{iu}^2 = r2^{t(k)}+2a^2 = N\mu_2$$

$$\sum x_{iu}^2 x_{ju}^2 = \lambda2^{t(k)} = N\mu_4$$

For the convenience N is replaced by M

By using the orthogonality condition we have

$$\mu_2^2 = \mu_4$$

$$\left(\frac{r2^{t(k)}+2a^2}{M}\right)^2 = \frac{\lambda2^{t(k)}}{M}$$

then we can obtain

$$a^2 = \left(\frac{\sqrt{\lambda2^{t(k)}M}-r2^{t(k)}}{2}\right)(\text{for orthogonality})$$

(3.1)

and the condition for the design will become rotatability is given bellow.

From equation 2 (ii) of (2.3) and 3 of (2.4), we have

$$\sum x_{iu}^4 = r2^{t(k)}+2a^4 = 3N\mu_4$$

$$\sum x_{iu}^2 x_{ju}^2 = \lambda 2^{t(k)} = N\mu_4$$

Then the rotatability condition equation (2.6), we have

$$\sum x_{iu}^4 = c \sum x_{iu}^2 x_{ju}^2$$

$$\Rightarrow r2^{t(k)} + 2a^4 = 3(\lambda 2^{t(k)})$$

$$\Rightarrow a^4 = \left(\frac{(3\lambda - r)2^{t(k)}}{2} \right) \text{(for rotatability)}$$

$$(3.2)$$

3.1 Proposed method of SORD of second type using pairwise balanced design

Let $(v, b, r, k_1, k_2, .., k_p, \lambda)$ denote a parameters of PBD, $k = \sup(k_1, k_2, .., k_p)$ and $2^{t(k)}$ denote a fractional replication of 2^k in +1 or -1 levels in which no interaction with less than five factors are confounded. $[1 - (v, b, r, k_1, k_2, ..., k_p, \lambda)]$ denote the design points generated from transpose of the incidence matrix of PBD. Let $[1 - (v, b, r, k_1, k_2, ..., k_p, \lambda)]2^{t(k)}$ are the $b2^{t(k)}$ design points generated from PBD by "multiplication" (cf. Raghavarao [30], pp 298-300). We use the additional set of points like $(\pm a_1, 0, ..., 0), (0, \pm a_1, 0, ..., 0), ..., (0, 0, ..., \pm a_1)$ and $(\pm a_2, 0, ..., 0), (0, \pm a_2, 0, ..., 0), ..., (0, 0, ..., \pm a_2)$ are two sets of axial points. Here $(a_1, 0, 0, ..., 0)2^1 U(a_2, 0, 0, ..., 0)2^1$ denote the $4v$ design points generated from $(a_1, 0, 0, ..., 0) \& (a_2, 0, 0, ..., 0)$ point sets. Let U denote the union of the design points generated from different sets of points, and (n_0) denote the number of central points. The method of construction of SORD of second type using PBD is given in the following theorem.

Theorem 1: The design points,

$$[1 - (v, b, r, k_1, k_2, .., k_p, \lambda)]2^{t(k)} U(a_1, 0, ..., 0)2^1 U(a_2, 0, ..., 0)2^1 U(n_0)$$

will give a v-dimensional SORD of second type using PBD in $N = b2^{t(k)} + 4v + n_0$ design points, with

$$a_1^4 + a_2^4 = \frac{2^{t(k)}(3\lambda - r)}{2} \text{ (for rotatability)}.$$

Proof: For the design points generated from second order rotatable designs of second type using PBD, simple symmetry conditions (2.2) are true. Further, conditions (2.3) and (2.4) are true as follows:

$$\sum x_{iu}^2 = r2^{t(k)} + 2(a_1^2 + a_2^2) = N\mu_2 \tag{3.3}$$

$$\sum x_{iu}^4 = r2^{t(k)} + 2(a_1^4 + a_2^4) = cN\mu_4$$
$$(3.4)$$

$$\sum x_{iu}^2 x_{ju}^2 = \lambda 2^{t(k)} = N\mu_4 \tag{3.5}$$

Solving equations (3.4) and (3.5), we get $a_1^4 + a_2^4 = \dfrac{2^{t(k)}(3\lambda\text{-r})}{2}$ (for rotatability).

Example 1: We illustrate the theorem 1 to obtain SORD of second type using PBD with parameters $(v=9, b=11, r=5, k_1=5, k_2=4, k_3=3, \lambda=2)$. The design points $[1\text{-}(9,11,5,5,4,3,2)]2^{t(5)}U(a_1,0,0,...,0)2^1U(a_2,0,0,...,0)2^1U(n_0=1)$ will give a v-dimensional SORD of second type using PBD in N=213 design points with one central point. From (3.3), (3.4) and (3.5) we have

$$\sum x_{iu}^2 = 80 + 2(a_1^2 + a_2^2) = N\mu_2 \tag{3.6}$$

$$\sum x_{iu}^4 = 80 + 2(a_1^4 + a_2^4) = cN\mu_4 \tag{3.7}$$

$$\sum x_{iu}^2 x_{ju}^2 = 32 = N\mu_4 \tag{3.8}$$

From (3.7) and (3.8), we can obtain the rotatability value $a_1^4 + a_2^4 = 8$. Here we assume for an arbitrary value $a_1 = 1$, then we get $a_2 = 1.6266$ and $c=3$ (for rotatability).

The non-singularity condition (2.5), we have

$$\frac{0.1502}{0.1502} > \frac{9}{3+9\text{-}1} \Rightarrow 1.0000 > 0.8182.$$

Hence the non singularity condition is also satisfied.

The variances and covariances of the estimated parameters are

$$V(\hat{b}_0) = 0.0258\sigma^2$$

$$V(\hat{b}_i) = 0.0121\sigma^2$$

$$V(\hat{b}_{ij}) = 0.0313\sigma^2$$

$$V(\hat{b}_{ii}) = 0.0156\sigma^2$$

$$Cov(\hat{b}_0, \hat{b}_{ii}) = \text{-}0.0061\sigma^2$$

$$Cov(\hat{b}_{ii}, \hat{b}_{jj}) = 0 \tag{3.9}$$

The variance of the estimated response at the point $(x_{10}, x_{20},...,x_{v0})$ is

$$V(\hat{Y}) = 0.0258\sigma^2 + d^2(\text{-}0.0001\sigma^2) + d^4(0.0156\sigma^2) \tag{3.10}$$

Table 1 gives the values of the variance of the estimated responses for different factors   using PBD.

Table 1 The variance of estimated response for different factors for $9 \leq v \leq 14$.

$(v, b, r, k_1, k_2, \ldots, k_p, \lambda)$	n_0	N	$a_1^4 + a_2^4$	a_1	a_2	$V(\hat{Y})$
(9,11,5,5,4,3,2)	1	213	8	1	1.6266	$(0.0258 - 0.0001\ d^2 + 0.0156\ d^4)\,\sigma^2$
(10,11,5,5,4,2)	1	217	8	1	1.6266	$(0.0276 + 0.0002\ d^2 + 0.0156\ d^4)\,\sigma^2$
(13,15,7,7,6,5,3)	34	1046	64	1	2.8173	$(0.0072 + 0.0026\ d^4)\,\sigma^2$
(14,15,7,7,6,3)	30	1046	64	1	2.8173	$(0.0076 + 0.0026\ d^4)\,\sigma^2$

4 Study of Orthogonality in SORD of Second Type Using PBD

An orthogonal design is one in which the terms in the fitted model are uncorrelated with one another and thus the parameter estimates are uncorrelated. In this case, the variance of the predicated response at any point x in the experimental region, is expressible as a weighted sum of the variance of the parameter estimates in the model. For second order moments $\sum x_{iu}^2$ and $\sum x_{iu}^2 x_{ju}^2$ is difficult to obtain. This is because the moments $\sum x_{iu}^2$ and $\sum x_{iu}^2 x_{ju}^2$ are necessarily positive. Hence, we consider the model with the pure quadratic terms correlated for their means. In regard to orthogonality, this model is often used for the sake of simplicity in the calculation. A design is said to be orthogonal we shall investigate the restriction $\left(\sum x_{iu}^2\right)^2 = N \sum x_{iu}^2 x_{ju}^2$ i.e $\left(N\mu_2^2\right)^2 = N\left(N\mu_4\right)$ i.e $\mu_2^2 = \mu_4$ to get SORD of second type using PBD.

$$a_1^2 + a_2^2 = \frac{\sqrt{N(\lambda 2^{t(k)})} - r2^{t(k)}}{2} \tag{4.1}$$

It must be established the equation (4.1) makes SORD of second type using PBD an orthogonal system. However $N = b2^{t(k)} + 4v + n_0$, the value of (4.1) depends on v, n_0 and the design points of SORD of second type using PBD. The following table 2 gives the values of orthogonality of second order response surface methodology using various parameters of SORD of second type using PBD and central points (n_0), the value of '$a_1^2 + a_2^2$' makes orthogonal second order response surface designs by using SORD of second type using PBD.

Let $(v,b,r,k_1,k_2,..,k_p,\lambda)$ denote a parameters of PBD, $k = \sup(k_1,k_2,..,k_p)$ and $2^{t(k)}$ denote a fractional replication of 2^k in +1 or -1 levels in which no interaction with less than five factors are confounded. $[1-(v,b,r,k_1,k_2,\ldots,k_p,\lambda)]$ denote the design points generated from transpose of the incidence matrix of PBD. Let $[1-(v,b,r,k_1,k_2,\ldots,k_p,\lambda)]2^{t(k)}$ are the $b2^{t(k)}$ design points generated from PBD by "multiplication" (cf. Raghavarao [30], pp 298-300). We use the additional set of points like $(\pm a_1,0,\ldots,0),(0,\pm a_1,0,\ldots,0),\ldots,(0,0,\ldots,\pm a_1)$ and $(\pm a_2,0,\ldots,0),(0,\pm a_2,0,\ldots,0),\ldots,(0,0,\ldots,\pm a_2)$ of two sets of axial points. Here $(a_1,0,0,\ldots,0)2^1 U(a_2,0,0,\ldots,0)2^1$ denote the 4v design points generated from $(a_1,0,0,\ldots,0)U(a_2,0,0,\ldots,0)$ point set. Let U denote the union of the design points generated from different sets of points, (n_0) denote the number of central points. The method of study of orthogonality of SORD of second type using PBD is given in the following theorem.

Theorem 2: The design points,

$$[1\text{-}(v,b,r,k_1,k_2,...,k_p,\lambda)]2^{t(k)}U(a_1,0,...,0)2^1U(a_2,0,...,0)2^1U(n_0)$$

will give a v-dimensional SORD of second type using PBD in $N=b2^{t(k)}+4v+n_0$ design points, with

$$a_1^2+a_2^2=\frac{\sqrt{N(\lambda 2^{t(k)})}-r2^{t(k)}}{2}(\text{for orthogonality}).$$

Proof: For the design points generated from second order rotatable designs of second type using PBD, simple symmetry conditions (2.2) are true. Further, conditions (2.3) and (2.4) are true as follows

$$\sum x_{iu}^2=r2^{t(k)}+2(a_1^2+a_2^2)=N\mu_2 \tag{4.2}$$

$$\sum x_{iu}^4=r2^{t(k)}+2(a_1^4+a_2^4)=cN\mu_4 \tag{4.3}$$

$$\sum x_{iu}^2x_{ju}^2=\lambda 2^{t(k)}=N\mu_4 \tag{4.4}$$

Solving equations (4.2) and (4.4) using $\mu_2^2=\mu_4$, we get

$$\left(\frac{r2^{t(k)}+2(a_1^2+a_2^2)}{N}\right)^2=\frac{\lambda 2^{t(k)}}{N}$$

then we obtain

$$a_1^2+a_2^2=\frac{\sqrt{N(\lambda 2^{t(k)})}-r2^{t(k)}}{2}(\text{for orthogonality}).$$

Example 2: We illustrate the theorem 2 second order rotatable designs of second type using PBD with parameters $(v=9,b=11,r=5,k_1=5,k_2=4,k_3=3,\lambda=2)$. The design points $[1\text{-}(9,11,5,5,4,3,2)]2^{t(5)}U(a_1,0,0,...,0)2^1U(a_2,0,0,...,0)2^1U(n_0=1)$ will give a v-dimensional SORD of second type using PBD in N=213 design points with one central point. From (3.3), (3.4) and (3.5) we have

$$\sum x_{iu}^2=80+2(a_1^2+a_2^2)=N\mu_2 \tag{4.5}$$

$$\sum x_{iu}^4=80+2(a_1^4+a_2^4)=cN\mu_4 \tag{4.6}$$

$$\sum x_{iu}^2x_{ju}^2=32=N\mu_4 \tag{4.7}$$

From (4.5) and (4.7), using $\mu_2^2=\mu_4$, we can obtain the orthogonality value

$$\Rightarrow a_1^2 + a_2^2 = \frac{\sqrt{(213) \times (32)} - 80}{2}$$

=1.2795.

The values of orthogonality of SORD of second type using PBD for $9 \leq v \leq 14$ with central points are given in the following Table 2.

Table 2. Values of orthogonality of SORD of second type using PBD for $9 \leq v \leq 14$

(9,11,5,5,4,3,2)			(10,11,5,5,4,2)			(13,15,7,7,6,5,3)			(14,15,7,7,6,3)		
(n_0=1)	N	$a_1^2 + a_2^2$	(n_0=1)	N	$a_1^2 + a_2^2$	(n_0=34)	N	$a_1^2 + a_2^2$	(n_0=30)	N	$a_1^2 + a_2^2$
n_0	213	1.2795	n_0	217	1.6653	n_0	1046	0.0714	n_0	1046	0.0714
n_0+1	214	1.3763	n_0+1	218	1.7612	n_0+1	1047	0.1785	n_0+1	1047	0.1785
n_0+2	215	1.4729	n_0+2	219	1.8569	n_0+2	1048	0.2855	n_0+2	1048	0.2855
n_0+3	216	1.5692	n_0+3	220	1.9524	n_0+3	1049	0.3925	n_0+3	1049	0.3925
n_0+4	217	1.6653	n_0+4	221	2.0476	n_0+4	1050	0.4994	n_0+4	1050	0.4994

5 Efficiency Comparison for SORD of Second Type Using PBD with SORD of First Type Using PBD

In this section, SORD of second type using PBD is used on the basis for estimating a specific coefficient in the response surface model, SORD of second type using PBD is compared with SORD of first type using PBD. This comparison criterion is based on the precision at which the coefficient is estimated. It is consider as the numbers of experimental plots are required at same way.

For example in terms of estimating mixed quadratic coefficient b_{ij} ($i \neq j$), lets try to compare two experimental designs, D_1 and D_2. The number of experimental plots required in D_1 and D_2 are M and N respectively. The relative efficiency of D_1 and D_2 is given by the following equation (see Myers [31], section 7.2).

$$E\left(D_1 \Big/ D_2\right) = \frac{\{Var(b_{ij}) \, in \, D_2\} N}{\{Var(b_{ij}) \, in \, D_1\} M} \tag{5.1}$$

Where

$N = b2^{t(k)} + 4v + n_0$ (Design points in SORD of second type using PBD)

$M = b2^{t(k)} + 2v + m_0$ (Design points in SORD of first type using PBD)

In this case, in order to compare fairly, the experimental system should make the second product equal to value of $\dfrac{\sum x_{iu}^2}{N}$. It must be scaled and for this the following scaling criteria will follows.

$$(i) \frac{1}{N} \sum x_{iu} = 0$$

$$(ii) \frac{1}{N} \sum x_{iu}^2 = 1, \, (i=1,2,...,v) \tag{5.2}$$

5.1 Comparison in mixed quadratic coefficient b_{ij} $(i \neq j)$

According to equation (2.7), $V(\hat{b}_{ij}) = \dfrac{\sigma^2}{N\mu_4}$, but this is before scaling equation (5.2) will be write in SORD of

second type using PBD. From (ii) of (5.2) $\dfrac{(r2^{t(k)} + 2a_1^2 + 2a_2^2)}{N}$ each time making equation (5.2) will be is equal

to 1, i.e. (ii) =1, we will written as the scaling factor g

i.e. $g = \left\{ \left(\dfrac{b2^{t(K)} + 4v + n_0}{r2^{t(k)} + 2a_1^2 + 2a_2^2} \right) \right\}^{\tfrac{1}{2}}$

$$(5.3)$$

However the $V(b_{ij})$ is multiplied by with the scaling factor 'g' than the $V(b_{ij}) = \dfrac{\sigma^2}{N\mu_4} \cdot \dfrac{1}{g^4}$ that is

$$V(b_{ij}) = \frac{\sigma^2}{N\mu_4} \left\{ \left[\frac{r2^{t(k)} + 2a_1^2 + 2a_2^2}{b2^{t(k)} + 4v + n_0} \right] \right\}^2$$

According to equation (5.1) the relative efficiency SORD of second type using PBD versus SORD of first type using PBD in the mixed quadratic coefficient b_{ij} is obtained as follows

$$E\left(\frac{\text{SORD of second type using PBD}}{\text{SORD of first type using PBD}} \right) = \frac{\dfrac{\sigma^2}{N\mu_4} \left(\dfrac{r2^{t(k)} + 2a^2}{b2^{t(k)} + 2v + m_0} \right)^2 (b2^{t(k)} + 2v + m_0)}{\dfrac{\sigma^2}{N\mu_4} \left(\dfrac{r2^{t(k)} + 2a_1^2 + 2a_2^2}{b2^{t(k)} + 4v + n_0} \right)^2 (b2^{t(k)} + 4v + n_0)}$$

$$= \frac{(r2^{t(k)} + 2a^2)^2 (b2^{t(k)} + 4v + n_0)}{(r2^{t(k)} + 2a_1^2 + 2a_2^2)^2 (b2^{t(k)} + 2v + m_0)} \tag{5.4}$$

From equation (5.4) the condition that $E\left(\dfrac{\text{SORD of second type using PBD}}{\text{SORD of first type using PBD}} \right) > 1$, than the SORD of second type using PBD is more efficient than SORD of first type using PBD.

$$a_1^2 + a_2^2 < \frac{1}{2} \left\{ (r2^{t(k)} + 2a^2) \sqrt{\frac{b2^{t(k)} + 4v + n_0}{b2^{t(k)} + 2v + m_0}} - r2^{t(k)} \right\} \tag{5.5}$$

From the values of (3.1) and (4.1) substitute in (5.4) and then we get the value of greater than 1. From this orthogonal SORD of second type using PBD have the same degree of efficiency as orthogonal SORD of first type using PBD, and consider the efficiency of SORD of second type using PBD is giving the better efficiency than SORD of first type using PBD. Now, the efficiency comparison of SORD of second type using PBD versus SORD of first type using PBD with rotatability. Substituting the values of (3.2) into (5.5) and we evaluated that the SORD of second type using PBD will be more efficient than the previous SORD of first type using PBD with rotatability.

$$a_1^2 + a_2^2 < \frac{1}{2} \left\{ \left(r2^{t(k)} + 2\sqrt{\frac{(3\lambda - r)2^{t(k)}}{2}} \right) \sqrt{\frac{b2^{t(k)} + 4v + n_0}{b2^{t(k)} + 2v + m_0}} - r2^{t(k)} \right\}$$

(5.6)

For example, in SORD of first type using PBD when $(v=9, b=11, r=5, k_1=5, k_2=4, k_3=3, \lambda=2)$ and $m_0=1$ then we get M =195 then a=1.6818 and $n_0=1$, then the equation (5.6)

$$a_1^2 + a_2^2 < 4.7599$$

(5.7)

Among the rotatability of SORD of second type using PBD, it is easy to find an experimental plan that satisfies the equation (5.7). For example in SORD of second type using PBD with $a_1=1$, $a_2=1.6266$ satisfy the rotatability property equation (3.7), and $a_1^2 + a_2^2 = 3.6458$ as it satisfies the equation (5.7) as well, it is more efficient than the rotatability SORD of first type using PBD. Then the relative efficiency of SORD of second type using PBD versus SORD of first type using PBD equation (5.4) is as follows.

$$\frac{(80 + 2(2.8285))^2 (176 + 36 + 1)}{(80 + 2(3.6458))^2 (176 + 18 + 1)} = 1.0518$$

5.2 Comparison in the pure quadratic coefficient b_{ii}

Now this time in terms of estimating the pure quadratic coefficient b_{ii}, the efficiency of SORD of second type using PBD is comparing with SORD of first type using PBD, here the scaling factor the equation (5.2) is applied. The relative efficiency SORD of second type using PBD versus SORD of first type using PBD is as follow based on the equation (5.1)

$$E\left(\frac{\text{SORD of second type using PBD}}{\text{SORD of first type using PBD}} \right) = \frac{\sigma^2 e_1 \left(\frac{r2^{t(k)} + 2a^2}{b2^{t(k)} + 2v + m_0} \right)^2 \left(b2^{t(k)} + 2v + m_0 \right)}{\sigma^2 e_2 \left(\frac{r2^{t(k)} + 2a_1^2 + 2a_2^2}{b2^{t(k)} + 4v + n_0} \right)^2 \left(b2^{t(k)} + 4v + n_0 \right)}$$

$$= \frac{e_1 \left(r2^{t(k)} + 2a^2 \right)^2 \left(b2^{t(k)} + 4v + n_0 \right)}{e_2 \left(r2^{t(k)} + 2a_1^2 + 2a_2^2 \right)^2 \left(b2^{t(k)} + 2v + m_0 \right)}$$

(5.8)

Where,

$e_1 = v(b_{ii})$ in SORD of first type using PBD

and

$e_2 = v(b_{ii})$ in SORD of second type using PBD.

For example, in SORD of second type using PBD of design $(v=9, b=11, r=5, k_1=5, k_2=4, k_3=3, \lambda=2)$, $n_0=1$, $a_1=1$, $a_2=1.6266$, $e_2=0.0156$ and in SORD of first type using PBD $m_0=1$, a=1.6818, $e_1= 0.0174$, lets us

compare the relative efficiency of SORD of second type using PBD versus SORD of first type using PBD, if you get the equation (5.8) as 1.1731, then we conclude that the SORD of second type using PBD is more efficient than SORD of first type using PBD.

5.3 Comparison in terms estimating the first order coefficient b_i

It can be developed in the same process of the $V(b_i)$ by multiplying the scaling factor then $V(b_i)=\dfrac{\sigma^2}{(r2^{t(k)}+2a_1^2+2a_2^2)}$ is to be multiplied by $1/g^2$ then we get, $V(b_i)=\dfrac{\sigma^2}{(r2^{t(k)}4v+n_0)}$ similarly the $V(b_{i)}$ is

multiplied by scaling factor in SORD of first type using PBD then we obtained $V(b_i)=\dfrac{\sigma^2}{(r2^{t(k)}+2v+n_0)}$ so

finally we compare the relative efficiency of $E\left(\dfrac{\text{SORD of second type using PBD}}{\text{SORD of first type using PBD}}\right)$ and it obtained 1, then the

efficiency of SORD of second type using PBD is more efficient than SORD of first type using PBD.

6 Conclusion

In this paper, SORD of second type using PBD is developed The variance covariance of the estimated parameters are studied and we evaluated for the SORD of second type using PBD is most orthogonal for second order response surface designs and the results of the orthogonality are also provided in the paper.

The comparison between the SORD of second type using PBD versus SORD of first type using PBD for different coefficients are studied then we conclude that the SORD of second type using PBD is more efficient than SORD of first type using PBD. It is convenient to use the practical situations and give the more efficiency when compared to SORD of first type using PBD.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Tyagi BN. On the construction of second order and third order rotatable designs through pairwise balanced designs and doubly balanced designs, Calcutta Statistical Association Bulletin. 1964;13:150-162.

[2] Box GEP, Hunter JS. Multifactor experimental designs for exploring response surfaces, Annals of Mathematical Statistics. 1957;28:195-241.

[3] Das MN, Narasimham VL. Construction of rotatable designs through balanced incomplete block designs, Annals of Mathematical Statistics. 1962;33:1421-1439.

[4] Raghavarao D. construction of second order rotatable designs through incomplete block designs, Journal of Indian Statistical Association. 1963;1:221-225.

[5] Draper NR, Guttman I. An index of rotatability, Technometrics. 1988;30:105-112.

[6] Khuri AI. A measure of rotatability for response surface designs, Technometrics. 1988;30:95-104.

[7]   Draper NR, Pukelsheim F. Another look at rotatability, Technometrics. 1990;32:195-202.

[8]   Park SH, Lim JH, Baba Y. A measure of rotatability for second order response surface designs, Annals of the Institute of Statistical Mathematics. 1993;45:655-664.

[9]   Das MN, Parasad R, Manocha VP. Response surface designs, symmetrical and asymmetrical, rotatable and modified, Statistics and Applications. 1999;1:17-34.

[10]   Kim Hyuk Joo. Extended central composite designs with the axial points indicated by two numbers, The Korean Communications in Statistics. 2002;9:595-605.

[11]   Victorbabu BRe, Vasundharadevi V. Modified second order response surface designs, rotatable designs using BIBD, Sri Lankan Journal of Applied Statistics. 2005;6:1-11.

[12]   Victorbabu BRe. Construction of modified second order rotatable and second order slope rotatable designs using a pair of balanced incomplete block designs, Sri Lankan Journal of Applied Statistics. 2006;7:39-53.

[13]   Victorbabu BRe. On second order rotatable designs – a review, International Journal of Agricultural and Statistical Sciences. 2007;3:201-209.

[14]   Victorbabu BRe, Vasundharadevi V, Viswanadam B. Modified second order response surface designs, using central composite designs, Canadian Journal of Pure and Applied Sciences. 2008;2:289-294.

[15]   Victorbabu BRe. Construction of modified second order rotatable designs using a pair of symmetrical unequal block arrangements with two unequal block sizes, Acharya Nagarjuna University: Journal of Physical Sciences. 2009;1:73-80.

[16]   Park HJ, Park SH. Extension of central composite designs for second order response surface model building, Communications in Statistics-Theory and Methods. 2010;39:1202-1211.

[17]   Victorbabu BRe, Surekha ChVVS. A note on measure of rotatability for second order response surface designs using incomplete block designs, Journal of Statistics: Advances in Theory and Applications. 2013;1:137-151.

[18]   Victorbabu BRe, Surekha ChVVS. A note on a measure of rotatability for second order response surface designs using balanced incomplete block designs, Thailand Statistician. 2015;13:97-110.

[19]   Victorbabu BRe, Jyostna P, Surekha ChVVS. Measure of rotatability for second order response surface designs using a pair of symmetrical unequal block arrangements with two unequal block sizes, International Journal of Agricultural and Statistical Sciences. 2016;12:9-11.

[20]   Victorbabu BRe. Jyostna P, Surekha ChVVS. Measure of rotatability for second order response surface designs using a pair of balanced incomplete block designs, Thailand Statistician. 2017;15:27-41.

[21]   Kim Hyuk Joo. On modified slope rotatability of central composite designs with two axial values, The Korean Journal of Applied Statistics. 2019;32:867-878.

[22]   Jyostna P. Victorbabu BRe. Evaluating measure of modified rotatability for second degree polynomial using balanced incomplete block designs, Asian Journal of Probability and Statistics. 2021a;10:47-59.

[23]   Jyostna P, Victorbabu BRe. Evaluating measure of modified rotatability for second degree polynomial using pairwise balanced designs, International Journal for Research in Applied Science and Engineering Technology. 2021b;9:506-514.

[24] Jyostna P, Victorbabu BRe. Measure of modified rotatability for second degree polynomial using symmetrical unequal block arrangements with two unequal block sizes, International Journal of Mathematics and Statistics Invention. 2021c;9:19-28.

[25] Jyostna P, Sulochana B, Victorbabu BRe. Measure of modified rotatability for second order response surface designs using central composite designs, Journal of Mathematical and Computational Science. 2021;11:494-519.

[26] Chiranjeevi P, Benhur KJ, Victorbabu BRe. Second order rotatable designs of second type using central composite designs, Asian Journal of Probability and Statistics. 2021;11:30-41.

[27] Chiranjeevi P, Victorbabu BRe. Second order rotatable designs of second type using symmetrical unequal block arrangements with two unequal block sizes, International Journal for Research in Applied Science and Engineering Technology. 2021a;9:515-525.

[28] Chiranjeevi P, Victorbabu BRe. Second order rotatable designs of second type using balanced incomplete Block Designs, In press; 2021b.

[29] Das MN, Giri NC. Design and Analysis of Experiments, New Age International (P) Limited, New Delhi-110 002, India; 1999.

[30] Raghavarao D. Constructions and combinatorial problems in Design of Experiments, John Wiley, New York; 1971.

[31] Myers RH. Response surface methodology, Blacksburg, VA: Author (Distributed by Edwards Brothers, Ann Arbor, MI); 1976.