Hindawi
Advances in Mathematical Physics
Volume 2021, Article ID 6630317, 8 pages
https://doi.org/10.1155/2021/6630317

Research Article

Uncertain Random Data Envelopment Analysis: Efficiency Estimation of Returns to Scale

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Received 30 November 2020; Revised 21 December 2020; Accepted 27 January 2021; Published 8 February 2021

Academic Editor: Mohammad Mirzazadeh

Evaluating efficiency according to the different states of returns to scale (RTS) is crucial to resource allocation and scientific decision for decision-making units (DMUs), but this kind of evaluation will become very difficult when the DMUs are in an uncertain random environment. In this paper, we attempt to explore the uncertain random data envelopment analysis approach so as to solve the problem that the inputs and outputs of DMUs are uncertain random variables. Chance theory is applied to handling the uncertain random variables, and hence, two evaluating models, one for increasing returns to scale (IRS) and the other for decreasing returns to scale (DRS), are proposed, respectively. Along with converting the two uncertain random models into corresponding equivalent forms, we also provide a numerical example to illustrate the evaluation results of these models.

1. Introduction

Data envelopment analysis (DEA) initiated by Charnes et al. [1], known as the CCR (Charnes, Cooper, and Rhodes) model, is one of the effective tools to evaluate efficiencies of DMUs with multiple inputs and multiple outputs. However, Banker [2] demonstrated that the CCR model only regarded that DMUs with constant returns to scale (CRS) were efficient. CRS is one of the states of returns to scale (RTS). Based on the RTS theory, RTS can be divided into three states as CRS, IRS (increasing returns to scale), and DRS (decreasing returns to scale) in accordance with the difference of output increment caused by input increment [3]. Subsequently, Banker et al. [4] proposed the BCC (Banker, Charnes, and Cooper) model to identify the efficient DMUs in the three states of RTS. The results revealed that the different states of RTS would affect the results of efficiency evaluation indeed. Afterwards, Fare and Grosskopf [5] refined the approach on measuring efficiencies of DMUs which exhibits DRS, and Seiford and Thrall [6] further estimated DMU's efficiency under IRS.

Along with the states of RTS affecting the results of efficiency evaluation, the inputs and outputs of DMUs that are not always observed accurately may affect the efficiency results as well. For example, early studies in DEA considered that such inputs and outputs as capital and labor are regarded as precise data. With more factors like carbon emission and social benefit taken into account in inputs and outputs nowadays, the traditional models are not suitable for dealing with these imprecise data. Then, some scholars regard these variables as random variables and treat them with probability theory. Therefore, many stochastic DEA models have been put forward including Li [7], Khodabakhshi et al. [8], and Cooper et al. [9].

However, some other scholars claim that these variables should be considered as uncertain variables because the uncertainty theory demonstrated that if the distribution function of a variable is not close enough to its real frequency, then it is better to treat it as an uncertain variable rather than a random variable [10]. Therefore, some uncertain DEA models are proposed via the application of uncertainty theory (Wen et al. [11], Lio and Liu [12], Jiang et al. [13], and Alireza and Lio [14]).

When the external environment becomes more complex, the imprecise inputs and outputs of DMUs may be not only a single random variable or uncertain variable but also both of them. In this case, some scholars attempt to take the uncertain random variables into account and propose uncertain random DEA models to estimate DMU's overall efficiency (Jiang et al. [15]) and technical efficiency (Jiang et al. [16]). However, a specific uncertain random model of examining the influence of RTS on efficiency evaluation does not exist currently. Motivated by this, this paper proposes two uncertain random models by applying chance theory [17] to dealing with uncertain random variables. One model is for estimating the DMUs' efficiency under IRS, and the other one is for DRS.

The remainder of this article is organized as follows. The second section will present a number of basic knowledge of uncertainty theory and chance theory. The third section will introduce the new uncertain random DEA model for IRS, and the equivalent form will be verified. The fourth section will introduce the new uncertain random DEA model for DRS, and the equivalent form will be verified as well. A numerical example to test two new uncertain random DEA models will be provided in the fifth section. The final section will make concluding remarks.

2. Preliminaries

In this part, we will briefly introduce the primary concepts and theorems of uncertainty theory and chance theory for the preparation to structure the new uncertain random DEA models in the next two sections.

2.1. Uncertainty Theory. As a powerful mathematical tool for dealing with uncertain variables and analyzing the belief degree, uncertainty theory was founded by Liu [10] in 2007. The uncertain measure M was defined as a set function on a σ-algebra $\mathscr{L}$ over a nonempty set Γ by the following axioms:

Axiom 1 (normality axiom). $M\{\Gamma\} = 1$ for the universal set Γ.

Axiom 2 (duality axiom). $M\{\Lambda\} + M\{\Lambda^c\} = 1$ for any event Λ.

Axiom 3 (subadditivity axiom). For every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$, we have

$$M\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} M\{\Lambda_i\}. \tag{1}$$

Then, Liu [18] proposed a product axiom in 2009.

Axiom 4 (product axiom). Let $(\Gamma_k, \mathscr{L}_k, M_k)$ be uncertainty spaces for $k = 1, 2, \cdots$. The product uncertain measure M is an uncertain measure satisfying

$$M\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} M_k\{\Lambda_k\}, \tag{2}$$

where Λ_k are arbitrarily chosen events from $\mathscr{L}_k$ for $k = 1, 2, \cdots$, respectively.

Definition 5 (Liu [10]). *An uncertain variable is a function ξ from an uncertainty space $(\Gamma, \mathscr{L}, M)$ to the set of real numbers such that $\{\xi \in B\}$ is an event for any Borel set B of real numbers.*

Definition 6 (Liu [19]). *An uncertain variable ξ is called linear if it has a linear uncertainty distribution*

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le a, \\ \dfrac{x-a}{b-a}, & \text{if } a < x \le b, \\ 1, & \text{if } x > b, \end{cases} \tag{3}$$

denoted by $\mathscr{L}(a, b)$ where a and b are real numbers with $a < b$.

Definition 7 (Liu [19]). *Let ξ be an uncertain variable with regular uncertainty distribution $\Phi(x)$. Then, the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ.*

Theorem 8 (Liu [19]). *Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(x_1, x_2, \cdots, x_n)$ is continuous, strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \tag{4}$$

has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f\left(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)\right). \tag{5}$$

Theorem 9 (Liu and Ha [20]). *Assume $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(\xi_1, \xi_2, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$, then*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \tag{6}$$

has an expected value

$$E[\xi] = \int_0^1 f\left(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)\right) d\alpha. \tag{7}$$

Uncertainty theory was subsequently studied by many researchers over the past decades, and many scholars have used uncertainty theory to model dynamic systems with uncertainty.

2.2. Chance Theory. Chance theory was put forward by Liu [17] in 2013 for modeling a complex system with the coexistence of uncertainty and randomness. Some elementary features and properties on uncertain random variables are defined as follows.

Definition 10 (Liu [21]). *An uncertain random variable is a function ξ from a chance space $(\Gamma, \mathscr{L}, M) \times (\Omega, \mathscr{A}, \Pr)$ to the set of real numbers such that $\{\xi \in B\}$ is an event in $\mathscr{L} \times \mathscr{A}$ for any Borel set B of real numbers.*

Definition 11 (Liu [21]). *Let ξ be an uncertain random variable. Then, its chance distribution is defined by*

$$\Phi(x) = \mathrm{Ch}\{\xi \leq x\} \quad (8)$$

for any $x \in \mathfrak{R}$.

Theorem 12 (Liu [17]). *Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with regular uncertainty distributions $Y_1, Y_2, \cdots, Y_n$, respectively. Assume $f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$ is continuous, strictly increasing with respect to $\tau_1, \tau_2, \cdots, \tau_k$ and strictly decreasing with respect to $\tau_{k+1}, \tau_{k+2}, \cdots, \tau_n$. Then, the uncertain random variable*

$$\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n) \quad (9)$$

has a chance distribution

$$\Phi(x) = \int_{\mathfrak{R}^m} F(x; y_1, y_2, \cdots, y_m) d\Psi_1(y_1) d\Psi_2(y_2) \cdots d\Psi_m(y_m), \quad (10)$$

where $F(x; y_1, y_2, \cdots, y_m)$ is the root α of the equation

$$f(y_1, y_2, \cdots, y_m, Y_1^{-1}(\alpha), \cdots, Y_k^{-1}(\alpha), Y_{k+1}^{-1}(1-\alpha), \cdots, Y_n^{-1}(1-\alpha)) = x. \quad (11)$$

Theorem 13 (Liu [17]). *Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with regular uncertainty distributions $Y_1, Y_2, \cdots, Y_n$, respectively. If f is a measurable function, then*

$$\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n) \quad (12)$$

has an expected value

$$E[\xi] = \int_{\mathfrak{R}^m} G(y_1, y_2, \cdots, y_m) d\Psi_1(y_1) d\Psi_2(y_2) \cdots d\Psi_m(y_m), \quad (13)$$

where

$$G(y_1, y_2, \cdots, y_m) = E[f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n)] \quad (14)$$

is the expected value of the uncertain variable $f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n)$ for any real numbers $y_1, y_2, \cdots, y_m$ and is determined by $Y_1, Y_2, \cdots, Y_n$.

Theorem 14 (Liu [17]). *Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with regular uncertainty distributions $Y_1, Y_2, \cdots, Y_n$, respectively. If $f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$ is a continuous and strictly increasing function (or strictly decreasing function) with respect to $\tau_1, \cdots, \tau_n$, then the expected function*

$$E[f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)] \quad (15)$$

is equal to

$$\int_{\mathfrak{R}^m} \int_0^1 f(y_1, \cdots, y_m, Y_1^{-1}(\alpha), \cdots, Y_n^{-1}(\alpha)) d\alpha d\Psi_1(y_1) \cdots d\Psi_m(y_m). \quad (16)$$

Based on the knowledge above, the two new uncertain random DEA models will be created in the following section.

3. Uncertain Random DEA Model for IRS

When the inputs and outputs of DMUs cannot be observed precisely, some of them were regarded as random variables and treated by probability theory, while some others were regarded as uncertain variables and treated by uncertainty theory. But in a more complex environment, the coexistence of random variables and uncertain variables in DMUs may occur. Therefore, a new approach to deal with uncertain random variables in estimating efficiency is necessary.

Suppose the number of DMUs is r. For each k with $1 \leq k \leq r$, the kth DMU consumes a random input vector x_k and an uncertain input vector $\tilde{x}_k$ to produce a random output vector y_k and an uncertain output vector $\tilde{y}_k$. For each DMU $_k$, we artificially set the expected ratio of weighted outputs to weighted inputs which is always less than or equal to unity, i.e.,

$$E\left[\frac{v^T \tilde{y}_k + v^T y_k}{\tilde{u}^T \tilde{x}_k + u^T x_k}\right] \leq 1, \quad k = 1, 2, \cdots, r, \quad (17)$$

where $\tilde{u}$, u, $\tilde{v}$, and v are nonnegative weight vectors. Subject to constraint (17), only a DMU which has CRS can find out a set of favorable weights $(\tilde{u}^*, u^*, \tilde{v}^*, v^*)$ such that the expected ratio of this DMU reaches up to 1, by which a DMU can be regarded as efficiency. The reason is that the increment of inputs of the DMU which exhibits CRS is equal to that of outputs.

According to the RTS theory, if the proportionate increases in outputs are larger than the proportionate increases in inputs, then the state of increasing returns to scale (IRS) arises [3]. In order to clarify the influence of IRS on efficiency values, we artificially set a factor, denoted as w, to adjust the proportion difference among input increment and output increment caused by IRS, and then, the constraint (17) is modified to

$$E\left[\frac{v^T\tilde{y}_k + v^T y_k - w}{\tilde{u}^T\tilde{x}_k + u^T x_k}\right] \le 1, \quad k = 1, 2, \cdots, r, \tag{18}$$

where w is less than or equal to 0, i.e., $w \le 0$. The new constraint (18) allows a DMU which exhibits IRS to also find out a set of favorable weights ($\tilde{u}^*$, u^*, $\tilde{v}^*$, v^*) such that the expected ratio of this DMU reaches up to 1. In this way, the DMUs under IRS can be considered efficient as well. In order to verify if the target DMU, distinguished by subscript "o," is efficient under IRS, we may solve the following uncertain random DEA model:

$$\begin{cases} \max\limits_{\tilde{u},u,\tilde{v},v,w} \vartheta_{\text{IRS}} = E\left[\dfrac{v^T\tilde{y}_o + v^T y_o - w}{u^T\tilde{x}_o + u^T x_o}\right] \\ \text{subject to}: \\ E\left[\dfrac{v^T\tilde{y}_k + v^T y_k - w}{u^T\tilde{x}_k + u^T x_k}\right] \le 1, \quad k = 1, 2, \cdots, r, \\ u, u, v, v \ge 0, \\ w \le 0, \end{cases} \tag{19}$$

where $\tilde{x}_k$, $\tilde{y}_k$, x_k, and y_k are uncertain input vectors, uncertain output vectors, random input vectors, and random output vectors of DMU $_k$, $k = 1, 2, \cdots, r$, respectively; u, v, $\tilde{u}$, and $\tilde{v}$ are nonnegative weight vectors; and $w \le 0$.

Definition 15 (IRS efficiency). *DMU $_o$ is regarded IRS efficient if the optimal value ϑ^*_{IRS} of ((19)) reaches up to 1.*

Theorem 16. *Let uncertain variables $\tilde{x}_{k1}, \cdots, \tilde{x}_{kj}$, $\tilde{y}_{k1}, \cdots, \tilde{y}_{kn}$ be independent with uncertainty distributions $\tilde{Y}_{k1}, \cdots, \tilde{Y}_{kj}$, $\tilde{\Pi}_{k1}, \cdots, \tilde{\Pi}_{kn}$, and let random variables $x_{k1}, \cdots, x_{ki}$, $y_{k1}, \cdots$, y_{km} be independent with probability distributions $\Phi_{k1}, \cdots$, Φ_{ki}, $\Psi_{k1}, \cdots, \Psi_{km}$, $k = 1, 2, \cdots, r$, respectively. Then, the new uncertain random DEA model for IRS ((19)) can be indicated as follows:*

$$\begin{cases} \max\limits_{\tilde{u},u,\tilde{v},v,w} \vartheta_{IRS} = \displaystyle\int_{\mathfrak{R}^+_{m+i}}\int_0^1 \frac{\sum_{p=1}^m v_p z_{op} + \sum_{t=1}^n \tilde{v}_t \tilde{\Pi}^{-1}_{ot}(\alpha) - w}{\sum_{q=1}^i u_q h_{oq} + \sum_{s=1}^j \tilde{u}_s \tilde{Y}^{-1}_{os}(1-\alpha)} d\alpha d\mathbf{\Phi_o}(\mathbf{h_o}) d\mathbf{\Psi_o}(\mathbf{z_o}) \\ \textit{subject to}: \\ \displaystyle\int_{\mathfrak{R}^+_{m+i}}\int_0^1 \frac{\sum_{p=1}^m v_p z_{kp} + \sum_{t=1}^n \tilde{v}_t \tilde{\Pi}^{-1}_{kt}(\alpha) - w}{\sum_{q=1}^i u_q h_{kq} + \sum_{s=1}^j \tilde{u}_s \tilde{Y}^{-1}_{ks}(1-\alpha)} d\alpha d\mathbf{\Phi_k}(\mathbf{h_k}) d\mathbf{\Psi_k}(\mathbf{z_k}) \le 1, \\ k = 1, 2, \cdots, r, \\ \tilde{u} = (\tilde{u}_1, \tilde{u}_2, \cdots, \tilde{u}_j) \ge 0, \\ u = (u_1, u_2, \cdots, u_i) \ge 0, \\ \tilde{v} = (\tilde{v}_1, \tilde{v}_2, \cdots, \tilde{v}_n) \ge 0, \\ v = (v_1, v_2, \cdots, v_m) \ge 0, \\ w \le 0, \end{cases} \tag{20}$$

where

$$\begin{aligned} d\mathbf{\Phi_o}(\mathbf{h_o}) &= d\Phi_{o1}(h_{o1}), d\Phi_{o2}(h_{o2}) \cdots d\Phi_{oi}(h_{oi}), \\ d\mathbf{\Psi_o}(\mathbf{z_o}) &= d\Psi_{o1}(z_{o1}), d\Psi_{o2}(z_{o2}) \cdots d\Psi_{om}(z_{om}), \\ d\mathbf{\Phi_k}(\mathbf{h_k}) &= d\Phi_{k1}(h_{k1}), d\Phi_{k2}(h_{k2}) \cdots d\Phi_{ki}(h_{ki}), \\ d\mathbf{\Psi_k}(\mathbf{z_k}) &= d\Psi_{k1}(z_{k1}), d\Psi_{k2}(z_{k2}) \cdots d\Psi_{km}(h_{km}). \end{aligned} \tag{21}$$

The uncertainty distributions of $\tilde{x}_{o1}, \cdots, \tilde{x}_{oj}$, $\tilde{y}_{o1}, \cdots, \tilde{y}_{on}$ are $\tilde{Y}_{o1}, \cdots, \tilde{Y}_{oj}$ and $\tilde{\Pi}_{o1}, \cdots, \tilde{\Pi}_{on}$, and the probability distributions of $x_{o1}, \cdots, x_{oi}$, $y_{o1}, \cdots, y_{om}$ are $\Phi_{o1}, \cdots, \Phi_{oi}$, $\Psi_{o1}, \cdots$, Ψ_{om}, respectively.

Proof. Since the function $(v^T y_k + \tilde{v}^T\tilde{y}_k - w)/(u^T x_k + \tilde{u}^T\tilde{x}_k)$ is a measurable function for each k with $1 \le k \le r$, it follows from Theorem 13 and we can obtain

$$\xi = \frac{v^T y_k + \tilde{v}^T\tilde{y}_k - w}{u^T x_k + \tilde{u}^T\tilde{x}_k} \tag{22}$$

has an expected value

$$E[\xi] = \int_{\mathfrak{R}^+_{m+i}} G(h_{k1}, \cdots, h_{ki}, z_{k1}, \cdots, z_{km}) d\mathbf{\Phi_k}(\mathbf{h_k}) \mathrm{d}\mathbf{\Psi_k}(\mathbf{z_k}) \tag{23}$$

for $k = 1, 2, \cdots, r$, where

$$\begin{aligned} G(h_{k1}, \cdots, h_{ki}, z_{k1}, \cdots, z_{km}) &= E\left[\frac{v^T z_k + \tilde{v}^T\tilde{y}_k - w}{u^T h_k + \tilde{u}^T\tilde{x}_k}\right], \\ d\mathbf{\Phi_k}(\mathbf{h_k}) &= d\Phi_{k1}(h_{k1}) d\Phi_{k2}(h_{k2}) \cdots d\Phi_{ki}(h_{ki}), \\ d\mathbf{\Psi_k}(\mathbf{z_k}) &= d\Psi_{k1}(z_{k1}) d\Psi_{k2}(z_{k2}) \cdots d\Psi_{km}(z_{km}), \end{aligned} \tag{24}$$

$k = 1, 2, \cdots, r$.

For each k with $1 \le k \le r$, since the function $(v^T z_k + \tilde{v}^T \tilde{y}_k - w)/(u^T h_k + \tilde{u}^T\tilde{x}_k)$ is strictly increasing with respect to $\tilde{y}_k$ and strictly decreasing with respect to $\tilde{x}_k$, by using Theorem 8, we can get the inverse uncertainty distribution is

$$R_k^{-1}(\alpha) = \frac{\sum_{p=1}^m v_p z_{kp} + \sum_{t=1}^n \tilde{v}_t \tilde{\Pi}^{-1}_{kt}(\alpha) - w}{\sum_{q=1}^i u_q h_{kq} + \sum_{s=1}^j \tilde{u}_s \tilde{Y}^{-1}_{ks}(1-\alpha)}. \tag{25}$$

Moreover, from Theorem 14, we can obtain

$$E\left[\frac{v^T z_k + \tilde{v}^T\tilde{y}_k - w}{u^T h_k + \tilde{u}^T\tilde{x}_k}\right] = \int_0^1 \frac{\sum_{p=1}^m v_p z_{kp} + \sum_{t=1}^n \tilde{v}_t \tilde{\Pi}^{-1}_{kt}(\alpha) - w}{\sum_{q=1}^i u_q h_{kq} + \sum_{s=1}^j \tilde{u}_s \tilde{Y}^{-1}_{ks}(1-\alpha)} d\alpha, \tag{26}$$

$k = 1, 2, \cdots, r$. Then, the equivalent form of equation (23) is

$$E[\xi] = \int_{\Re^+_{m+i}} \int_0^1 \frac{\sum_{p=1}^m v_p z_{kp} + \sum_{t=1}^n \tilde{v}_t \tilde{\Pi}_{kt}^{-1}(\alpha) - w}{\sum_{q=1}^i u_q h_{kq} + \sum_{s=1}^j \tilde{u}_s \tilde{Y}_{ks}^{-1}(1-\alpha)} d\alpha \mathbf{\Phi_k}(\mathbf{h_k}) \mathrm{d}\mathbf{\Psi_k}(\mathbf{z_k}), \tag{27}$$

where

$$\begin{aligned} d\mathbf{\Phi_k}(\mathbf{h_k}) &= d\Phi_{k1}(h_{k1}) d\Phi_{k2}(h_{k2}) \cdots d\Phi_{ki}(h_{ki}), \\ d\mathbf{\Psi_k}(\mathbf{z_k}) &= d\Psi_{k1}(z_{k1}) d\Psi_{k2}(z_{k2}) \cdots d\Psi_{km}(z_{km}), \end{aligned} \tag{28}$$

$k = 1, 2, \cdots, r$. The proof is completed.

4. Uncertain Random DEA Model for DRS

In this part, we propose the uncertain random DEA model for DRS. According to RTS theory, if the proportionate increases in outputs are smaller than the proportionate increases in inputs, then the state of decreasing returns to scale (DRS) prevails [3]. Then, the constraint (17) is modified to

$$E\left[\frac{v^T \tilde{y}_k + v^T y_k - w}{\tilde{u}^T \tilde{x}_k + u^T x_k}\right] \le 1, \quad k = 1, 2, \cdots, r, \tag{29}$$

where w is greater than or equal to 0, i.e., $w \ge 0$. The new constraint (29) allows a DMU which exhibits DRS to also find out a set of favorable weights $(\tilde{u}^*, u^*, \tilde{v}^*, v^*)$ such that the expected ratio of this DMU reaches up to 1. In this way, the DMUs under DRS can be considered efficient as well. We still distinguish target DMU by subscript "o," then verify if it is efficient under DRS, and may solve the following uncertain random DEA model:

$$\begin{cases} \max\limits_{\tilde{u},u,\tilde{v},v,w} \vartheta_{\mathrm{DRS}} = E\left[\dfrac{v^T \tilde{y}_o + v^T y_o - w}{u^T \tilde{x}_o + u^T \mathrm{x}_o}\right] \\ \text{subject to :} \\ E\left[\dfrac{v^T \tilde{y}_k + v^T y_k - w}{u^T \tilde{x}_k + u^T x_k}\right] \le 1, \quad k = 1, 2, \cdots, r, \\ u, u, v, v \ge 0, \\ w \ge 0, \end{cases} \tag{30}$$

where $\tilde{x}_k$, $\tilde{y}_k$, x_k, and y_k are uncertain input vectors, uncertain output vectors, random input vectors, and random output vectors of DMU_k, $k = 1, 2, \cdots, r$, respectively; u, v, $\tilde{u}$, and $\tilde{v}$ are nonnegative weight vectors; and $w \ge 0$.

Definition 17 (DRS efficiency). *DMU $_o$ is regarded DRS efficient if the optimal value $\vartheta^*_{\mathrm{DRS}}$ of ((30)) reaches up to 1.*

Theorem 18. *Let uncertain variables $\tilde{x}_{k1}, \cdots, \tilde{x}_{kj}$, $\tilde{y}_{k1}, \cdots, \tilde{y}_{kn}$ be independent with uncertainty distributions $\tilde{Y}_{k1}, \cdots, \tilde{Y}_{kj}$, $\tilde{\Pi}_{k1}, \cdots, \tilde{\Pi}_{kn}$, and let random variables $x_{k1}, \cdots, x_{ki}$, $y_{k1}, \cdots$, y_{km} be independent with probability distributions $\Phi_{k1}, \cdots$, Φ_{ki}, $\Psi_{k1}, \cdots, \Psi_{km}$, $k = 1, 2, \cdots, r$, respectively. Then, the new uncertain random DEA model for DRS ((30)) can be indicated as follows:*

$$\begin{cases} \max\limits_{\tilde{u},u,\tilde{v},v,w} \vartheta_{DRS} = \displaystyle\int_{\Re^+_{m+i}} \int_0^1 \frac{\sum_{p=1}^m v_p z_{op} + \sum_{t=1}^n \tilde{v}_t \tilde{\Pi}_{ot}^{-1}(\alpha) - w}{\sum_{q=1}^i u_q h_{oq} + \sum_{s=1}^j \tilde{u}_s \tilde{Y}_{os}^{-1}(1-\alpha)} d\alpha d\mathbf{\Phi_o}(\mathbf{h_o}) d\mathbf{\Psi_o}(\mathbf{z_o}) \\ \text{subject to :} \\ \displaystyle\int_{\Re^+_{m+i}} \int_0^1 \frac{\sum_{p=1}^m v_p z_{kp} + \sum_{t=1}^n \tilde{v}_t \tilde{\Pi}_{kt}^{-1}(\alpha) - w}{\sum_{q=1}^i u_q h_{kq} + \sum_{s=1}^j \tilde{u}_s \tilde{Y}_{ks}^{-1}(1-\alpha)} d\alpha d\mathbf{\Phi_k}(\mathbf{h_k}) d\mathbf{\Psi_k}(\mathbf{z_k}) \le 1, \\ k = 1, 2, \cdots, r, \\ \tilde{u} = (\tilde{u}_1, \tilde{u}_2, \cdots, \tilde{u}_j) \ge 0, \\ u = (u_1, u_2, \cdots, u_i) \ge 0, \\ \tilde{v} = (\tilde{v}_1, \tilde{v}_2, \cdots, \tilde{v}_n) \ge 0, \\ v = (v_1, v_2, \cdots, v_m) \ge 0, \\ w \ge 0, \end{cases} \tag{31}$$

where

$$\begin{aligned} d\mathbf{\Phi_o}(\mathbf{h_o}) &= d\Phi_{o1}(h_{o1}), d\Phi_{o2}(h_{o2}) \cdots d\Phi_{oi}(h_{oi}), \\ d\mathbf{\Psi_o}(\mathbf{z_o}) &= d\Psi_{o1}(z_{o1}), d\Psi_{o2}(z_{o2}) \cdots d\Psi_{om}(z_{om}), \\ d\mathbf{\Phi_k}(\mathbf{h_k}) &= d\Phi_{k1}(h_{k1}), d\Phi_{k2}(h_{k2}) \cdots d\Phi_{ki}(h_{ki}), \\ d\mathbf{\Psi_k}(\mathbf{z_k}) &= d\Psi_{k1}(z_{k1}), d\Psi_{k2}(z_{k2}) \cdots d\Psi_{km}(h_{km}). \end{aligned} \tag{32}$$

The uncertainty distributions of $\tilde{x}_{o1}, \cdots, \tilde{x}_{oj}$, $\tilde{y}_{o1}, \cdots, \tilde{y}_{on}$ are $\tilde{Y}_{o1}, \cdots, \tilde{Y}_{oj}$ and $\tilde{\Pi}_{o1}, \cdots, \tilde{\Pi}_{on}$, and the probability distributions of $x_{o1}, \cdots, x_{oi}$, $y_{o1}, \cdots, y_{om}$ are $\Phi_{o1}, \cdots, \Phi_{oi}$, $\Psi_{o1}, \cdots$, Ψ_{om}, respectively.

Proof. Since the function $(v^T y_k + \tilde{v}^T \tilde{y}_k - w)/(u^T x_k + \tilde{u}^T \tilde{x}_k)$ is a measurable function for each k with $1 \le k \le r$, it follows from Theorem 13 and we can obtain

$$\varsigma = \frac{v^T y_k + \tilde{v}^T \tilde{y}_k - w}{u^T x_k + \tilde{u}^T \tilde{x}_k} \tag{33}$$

has an expected value

$$E[\varsigma] = \int_{\Re^+_{m+i}} G(h_{k1}, \cdots, h_{ki}, z_{k1}, \cdots, z_{km}) d\mathbf{\Phi_k}(\mathbf{h_k}) d\mathbf{\Psi_k}(\mathbf{z_k}) \tag{34}$$

for $k = 1, 2, \cdots, r$, where

$$\begin{aligned} G(h_{k1}, \cdots, h_{ki}, z_{k1}, \cdots, z_{km}) &= E\left[\frac{v^T z_k + \tilde{v}^T \tilde{y}_k - w}{u^T h_k + \tilde{u}^T \tilde{x}_k}\right], \\ d\mathbf{\Phi_k}(\mathbf{h_k}) &= d\Phi_{k1}(h_{k1}) d\Phi_{k2}(h_{k2}) \cdots d\Phi_{ki}(h_{ki}), \\ d\mathbf{\Psi_k}(\mathbf{z_k}) &= d\Psi_{k1}(z_{k1}) d\Psi_{k2}(z_{k2}) \cdots d\Psi_{km}(z_{km}), \end{aligned} \tag{35}$$

$k = 1, 2, \cdots, r$.

Table 1: Fifteen DMUs with three inputs and three outputs.

DMU_k	Output variables			Input variables		
	$\tilde{y}_1$ (uncertain)	$\tilde{y}_2$ (uncertain)	y_3 (random)	$\tilde{x}_1$ (uncertain)	$\tilde{x}_2$ (uncertain)	x_3 (random)
1	$\mathscr{L}(51, 65)$	$\mathscr{L}(45, 55)$	$U(60, 78)$	$\mathscr{L}(8, 13)$	$\mathscr{L}(10, 15)$	$U(9, 14)$
2	$\mathscr{L}(64, 71)$	$\mathscr{L}(60, 69)$	$U(76, 88)$	$\mathscr{L}(17, 20)$	$\mathscr{L}(12, 19)$	$U(10, 16)$
3	$\mathscr{L}(55, 60)$	$\mathscr{L}(80, 96)$	$U(77, 89)$	$\mathscr{L}(10, 19)$	$\mathscr{L}(11, 20)$	$U(10, 21)$
4	$\mathscr{L}(47, 55)$	$\mathscr{L}(79, 95)$	$U(62, 80)$	$\mathscr{L}(15, 24)$	$\mathscr{L}(12, 21)$	$U(12, 23)$
5	$\mathscr{L}(49, 62)$	$\mathscr{L}(54, 67)$	$U(60, 75)$	$\mathscr{L}(10, 16)$	$\mathscr{L}(13, 19)$	$U(14, 17)$
6	$\mathscr{L}(46, 52)$	$\mathscr{L}(55, 68)$	$U(65, 76)$	$\mathscr{L}(11, 19)$	$\mathscr{L}(16, 22)$	$U(13, 19)$
7	$\mathscr{L}(52, 60)$	$\mathscr{L}(67, 80)$	$U(70, 83)$	$\mathscr{L}(13, 21)$	$\mathscr{L}(10, 17)$	$U(12, 18)$
8	$\mathscr{L}(54, 67)$	$\mathscr{L}(65, 75)$	$U(55, 69)$	$\mathscr{L}(11, 16)$	$\mathscr{L}(14, 20)$	$U(8, 16)$
9	$\mathscr{L}(45, 55)$	$\mathscr{L}(62, 76)$	$U(63, 78)$	$\mathscr{L}(10, 15)$	$\mathscr{L}(8, 12)$	$U(14, 21)$
10	$\mathscr{L}(48, 57)$	$\mathscr{L}(54, 63)$	$U(60, 73)$	$\mathscr{L}(11, 17)$	$\mathscr{L}(9, 18)$	$U(8, 17)$
11	$\mathscr{L}(56, 69)$	$\mathscr{L}(64, 76)$	$U(66, 83)$	$\mathscr{L}(8, 12)$	$\mathscr{L}(10, 20)$	$U(11, 18)$
12	$\mathscr{L}(60, 78)$	$\mathscr{L}(63, 70)$	$U(74, 83)$	$\mathscr{L}(6, 13)$	$\mathscr{L}(7, 15)$	$U(15, 20)$
13	$\mathscr{L}(58, 66)$	$\mathscr{L}(80, 97)$	$U(78, 90)$	$\mathscr{L}(5, 11)$	$\mathscr{L}(12, 25)$	$U(10, 17)$
14	$\mathscr{L}(47, 58)$	$\mathscr{L}(68, 78)$	$U(70, 88)$	$\mathscr{L}(12, 19)$	$\mathscr{L}(14, 22)$	$U(14, 25)$
15	$\mathscr{L}(44, 53)$	$\mathscr{L}(53, 69)$	$U(63, 72)$	$\mathscr{L}(14, 20)$	$\mathscr{L}(9, 15)$	$U(13, 21)$

For each k with $1 \le k \le r$, since the function $(v^T z_k + \tilde{v}^T \tilde{y}_k - w)/(u^T h_k + \tilde{u}^T \tilde{x}_k)$ is strictly increasing with respect to $\tilde{y}_k$ and strictly decreasing with respect to $\tilde{x}_k$, by using Theorem 8, we can get the inverse uncertainty distribution is

$$R_k^{-1}(\alpha) = \frac{\sum_{p=1}^{m} v_p z_{kp} + \sum_{t=1}^{n} \tilde{v}_t \tilde{\Pi}_{kt}^{-1}(\alpha) - w}{\sum_{q=1}^{i} u_q h_{kq} + \sum_{s=1}^{j} \tilde{u}_s \tilde{Y}_{ks}^{-1}(1-\alpha)}. \tag{36}$$

Moreover, from Theorem 14, we can obtain

$$E\left[\frac{v^T z_k + \tilde{v}^T \tilde{y}_k - w}{u^T h_k + \tilde{u}^T \tilde{x}_k}\right] = \int_0^1 \frac{\sum_{p=1}^{m} v_p z_{kp} + \sum_{t=1}^{n} \tilde{v}_t \tilde{\Pi}_{kt}^{-1}(\alpha) - w}{\sum_{q=1}^{i} u_q h_{kq} + \sum_{s=1}^{j} \tilde{u}_s \tilde{Y}_{ks}^{-1}(1-\alpha)} d\alpha, \tag{37}$$

$k = 1, 2, \cdots, r$. Then, the equivalent form of equation (34) is

$$E[\varsigma] = \int_{\Re_{m+i}^{+}} \int_0^1 \frac{\sum_{p=1}^{m} v_p z_{kp} + \sum_{t=1}^{n} \tilde{v}_t \tilde{\Pi}_{kt}^{-1}(\alpha) - w}{\sum_{q=1}^{i} u_q h_{kq} + \sum_{s=1}^{j} \tilde{u}_s \tilde{Y}_{ks}^{-1}(1-\alpha)} d\alpha \mathbf{\Phi}_{\mathbf{k}}(\mathbf{h}_{\mathbf{k}}) d\mathbf{\Psi}_{\mathbf{k}}(\mathbf{z}_{\mathbf{k}}), \tag{38}$$

where

$$\begin{aligned} d\mathbf{\Phi}_{\mathbf{k}}(\mathbf{h}_{\mathbf{k}}) &= d\Phi_{k1}(h_{k1}) d\Phi_{k2}(h_{k2}) \cdots d\Phi_{ki}(h_{ki}), \\ d\mathbf{\Psi}_{\mathbf{k}}(\mathbf{z}_{\mathbf{k}}) &= d\Psi_{k1}(z_{k1}) d\Psi_{k2}(z_{k2}) \cdots d\Psi_{km}(z_{km}), \end{aligned} \tag{39}$$

$k = 1, 2, \cdots, r$. The proof is completed.

5. A Numerical Example

In order to examine the two new uncertain random DEA models, this section presents fifteen DMUs with three inputs and three outputs to demonstrate an illustrative example. Among these inputs and outputs, two inputs and two outputs are uncertain variables subject to linear uncertainty distributions represented as $\mathscr{L}(a, b)$, and one input and one output are random variables subject to uniform distributions represented as $U(a, b)$. The original data of these DMUs are provided in Table 1.

According to the data in Table 1, we can obtain each DMU's IRS efficiency by calculating the optimal value ϑ_{IRS}^* of model (19) and DRS efficiency by calculating the optimal value ϑ_{DRS}^* of model (30). In addition, to further clarify the influence of RTS on efficiency values, we have also examined two other cases, one for overall efficiency and the other for technical efficiency. If $w = 0$, overall efficiencies of DMUs can be calculated, represented as η^*, and technical efficiencies of DMUs, represented as φ^*, can be gained under the condition that w is unconstrained in sign. The results of the four kinds of efficiencies of each DMU are shown in Table 2.

As shown in Table 2, the second column represents DMUs' IRS efficiencies. It is obvious that the five DMUs are IRS efficient because the optimal values ϑ_{IRS}^* of them reach up to 1, and the other ten are IRS inefficient. Similarly, four DMUs are DRS efficient shown in the third column. Among these efficient DMUs, some like DMU_3, DMU_8, and DMU_9 are IRS efficient but DRS inefficient, while some like DMU_1 and DMU_2 are DRS efficient but IRS inefficient. However, there are also two DMUs (DMU_{12} and DMU_{13}) that are both IRS efficient and DRS efficient. Does it mean that these two DMUs are overall efficient as well?

Table 2: The efficiency evaluation results of different uncertain random DEA models.

DMU_k	Optimal values			
	ϑ^*_{IRS}	ϑ^*_{DRS}	η^*	φ^*
1	0.9702	1.0000	0.4366	1.0000
2	0.9745	1.0000	0.5699	1.0000
3	1.0000	0.8829	0.5678	1.0000
4	0.8784	0.8803	0.2686	0.9185
5	0.6766	0.4042	0.3251	0.8219
6	0.6584	0.3746	0.3433	0.6062
7	0.9542	0.4382	0.3451	0.9583
8	1.0000	0.5223	0.3661	1.0000
9	1.0000	0.8316	0.3600	1.0000
10	0.7588	0.4123	0.3535	0.9047
11	0.9569	0.7897	0.4859	0.9405
12	1.0000	1.0000	1.0000	1.0000
13	1.0000	1.0000	1.0000	1.0000
14	0.7667	0.7667	0.3275	0.7362
15	0.7525	0.3414	0.4764	0.6200

Overall efficiencies of DMUs are exhibited in the fourth column of Table 2, represented as η^*. It is clear that only two DMUs, DMU_{12} and DMU_{13}, can be regarded as overall efficient among fifteen DMUs. Therefore, it can be inferred that if a DMU prevails IRS efficiency and DRS efficiency simultaneously, it prevails overall efficiency as well. The result is in accordance with the assumption that the efficiency value is affected by the RTS state.

The last column shows DMUs' technical efficiencies, represented as φ^*. There are a total of seven DMUs that can be considered technical efficient. The number is just the aggregation of DMUs which are both IRS efficient and DRS efficient. This phenomenon demonstrates that the technical efficiency compounds three kinds of different efficiencies caused by RTS, although it cannot differ IRS efficiency and DRS efficiency.

6. Conclusions

In this paper, we introduced two new models, uncertain random DEA model for IRS and uncertain random DEA model for DRS, so as to focus on efficiency evaluation of DMUs under the uncertain random environment. Meanwhile, we presented the equivalent forms of the two new models and provided detailed proof processes. Finally, a numerical example was given to demonstrate the evaluation results of these two models. Considering the situation that uncertain variables and random variables coexist in the inputs and outputs of DMUs simultaneously, our work broadens the application of chance theory in efficiency evaluation in practice. In addition, this paper is also expected to be applied in the field of logistics in the future.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the National Natural Science Foundation of China Grant No. 61873329 and Research Program of Natural Science Foundation of Shandong Province Grant No. ZR2020MG044.

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