Research Article

Logarithmic Generalization of the Lambert W Function and Its Applications to Adiabatic Thermostatistics of the Three-Parameter Entropy

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A generalization of the Lambert W function called the logarithmic Lambert function is introduced and is found to be a solution to the thermostatistics of the three-parameter entropy of classical ideal gas in adiabatic ensembles. The derivative, integral, Taylor series, approximation formula, and branches of the function are obtained. The heat functions and specific heats are computed using the "unphysical" temperature and expressed in terms of the logarithmic Lambert function.

1. Introduction

The first law of thermodynamics [1] states that the total energy of a system remains constant, even if it is converted from one form to another. For example, kinetic energy—the energy that an object possesses when it moves—is converted to heat energy when a driver presses the brakes on the car to slow it down. The first law of thermodynamics relates the various forms of kinetic and potential energy in a system to the work, which a system can perform, and to the transfer of heat. This law is sometimes taken as the definition of internal energy and also introduces an additional state variable, enthalpy. The first law of thermodynamics allows for many possible states of a system to exist. However, experience indicates that only certain states occur. This eventually leads to the second law of thermodynamics and the definition of another state variable called entropy.

Entropy is a measure of the number of specific ways in which a thermodynamic system may be arranged, commonly understood as a measure of disorder. According to the second law of thermodynamics, the entropy of an isolated system never decreases; such a system will spontaneously proceed towards thermodynamic equilibrium, the configuration with maximum entropy [2].

A system in thermodynamic equilibrium with its surroundings can be described using three macroscopic variables corresponding to the thermal, mechanical, and chemical equilibrium. For each fixed value of these macroscopic variables (macrostates), there are many possible microscopic configurations (microstates). A collection of systems existing in the various possible microstates, but characterized by the same macroscopic variables, is called an ensemble. The adiabatic class has the heat function as its thermal equilibrium variable. The specific form of each of the four adiabatic ensembles and its heat function and corresponding entropy are listed in Table 1 (see [3]).

It is known that some physical systems cannot be described by Boltzmann-Gibbs (BG) statistical mechanics [4, 5]. Among these physical systems are diffusion [6], turbulence [7], transverse momentum distribution of hadron jets in $e^{+}e^{-}$ collisions [8], thermalization of heavy quarks in a collisional process [9], astrophysics [10], and solar neutrinos [11]. To overcome some difficulties in dealing with these systems, Tsallis [12] introduced a generalized entropic form, the

q-entropy:

$$S_q = k \sum_{i=1}^{\omega} p_i \ln_q \frac{1}{p_i}, \tag{1}$$

where k is a positive constant and ω is the total number of microscopic states.

For $q > 0$, $\ln_q x$ called the q-logarithm is defined as

$$\ln_q x = \frac{x^{1-q} - 1}{1 - q}, \qquad \ln_1 x = \ln x. \tag{2}$$

The inverse function of the q-logarithm is called q-exponential and is given by

$$\exp_q x = [1 + (1-q)x]^{1/(1-q)}, \qquad \exp_1 x = \exp x. \tag{3}$$

In the case of equiprobability, BG is recovered in the limit $q \longrightarrow 1$.

A two-parameter entropy $S_{q,q}{}'$ that recovered the q-entropy S_q in the limit $q' \longrightarrow 1$ was defined in [13] as

$$S_{q,q}{}'x \equiv \sum_{i=1}^{\omega} p_i \ln_{q,q'} \frac{1}{p_i} = \frac{1}{1-q'} \sum_{i=1}^{\omega} p_i \left[\exp \left(\frac{1-q'}{1-q} \left(p_i^{q-1} - 1 \right) \right) - 1 \right]. \tag{4}$$

Applications of S_q to a class of energy-based ensembles were done in [14] while applications of $S_{q,q}{}'$ to adiabatic ensembles were done in [3]. Results in the applications of $S_{q,q}{}'$ involved the well-known Lambert W function.

A three-parameter entropy $S_{q,q',r}$ that recovers $S_{q,q}{}'$ in the limit $r \longrightarrow 1$ was defined in [15] as

$$S_{q,q',r} \equiv k \sum_{i=1}^{\omega} p_i \ln_{q,q',r} \frac{1}{p_i}, \tag{5}$$

where k is a positive constant and

$$\ln_{q,q',r} x \equiv \frac{1}{1-r} \left(\exp \left(\frac{1-r}{1-q'} \left(e^{(1-q') \ln_q x} - 1 \right) - 1 \right) \right). \tag{6}$$

This three-parameter entropy was shown to be nonextensive (see [15]).

One of the interesting properties of entropy is the Lesche stability [16]. The Lesche stability of the κ-entropy was shown in [17].

It was shown in [15] that the three-parameter logarithm $\ln_{q,q',r} x$ is differentiable. Hence, the function

$$f(x) = x \ln_{q,q',r} \frac{1}{x}, \qquad x \neq 0 \tag{7}$$

is also differentiable. Since differentiability implies continuity, for every $\epsilon > 0$, there exists $\delta_i > 0$ such that if $|\rho_i - \lambda_i| <$

δ_i, then $|f(\rho_i) - f(\lambda_i)| < \epsilon |\ln_{q,q',r} \omega|/\omega$. Now, given

$$\rho \equiv \{\rho_i : i = 1, 2, \cdots, \omega\},$$
$$\lambda \equiv \{\lambda_i : i = 1, 2, \cdots, \omega\}, \tag{8}$$
$$\|\rho - \lambda\|_1 = \sum_{i=1}^{\omega} |\rho_i - \lambda_i| < \sum_{i=1}^{\omega} \frac{\delta_i}{\omega} \leq \delta,$$

where $\delta = \min \{\delta_1, \delta_2, \cdots, \delta_\omega\}$. This implies that

$$\left| S_{q,q',r}(\rho) - S_{q,q',r}(\lambda) \right| = \left| k \sum_{i=1}^{\omega} \rho_i \ln_{q,q',r} \frac{1}{\rho_i} - k \sum_{i=1}^{\omega} \lambda_i \ln_{q,q',r} \frac{1}{\lambda_i} \right|$$
$$\leq k \sum_{i=1}^{\omega} \left| \rho_i \ln_{q,q',r} \frac{1}{\rho_i} - \lambda_i \ln_{q,q',r} \frac{1}{\lambda_i} \right|$$
$$< k \sum_{i=1}^{\omega} \frac{\epsilon \left| \ln_{q,q',r} \omega \right|}{\omega} = k\epsilon \left| \ln_{q,q',r} \omega \right|. \tag{9}$$

Note that the maximum value of $S_{q,q',r}(\rho)$ corresponding to the uniform distribution $\rho \equiv \{\rho_i = 1/\omega : i = 1, 2, \cdots, \omega\}$ is given by (see [15])

$$S_{q,q',r}^{\max} = k \ln_{q,q',r} \omega. \tag{10}$$

Thus, we have

$$\left| \frac{S_{q,q',r}(\rho) - S_{q,q',r}(\lambda)}{S_{q,q',r}^{\max}} \right| < \epsilon. \tag{11}$$

Therefore, the three-parameter entropy is Lesche-stable. Moreover, it was also shown that the three-parameter entropy is concave and convex in specified ranges of the parameters (see [15]).

The goal of this paper is to introduce a generalized Lambert W function and derive its applications to the adiabatic thermostatistics of the three-parameter entropy of classical ideal gas. This generalized Lambert W function will be called the logarithmic Lambert function. As the logarithmic Lambert function is new, it is imperative that we study its analytic properties, namely, the derivative, integral, Taylor series, approximation, and branches of the function. These calculations are motivated by the hope to obtain results which are similar to those obtained for the quadratic Lambert function [18] which have applications to resolving the Einstein-Maxwell field equation with plane symmetry.

The properties of the logarithmic Lambert function have implications in the applications to the adiabatic thermostatistics of the three-parameter entropy of classical ideal gas. In particular, the derivative is useful to determine the branches of the function and the specific heat functions in the applications. In the computation of the heat functions and the specific heats, we used the inverse of the Lagrange multiplier for the temperature T which is referred to in [19] as the "unphysical" temperature.

The integral is being derived because when the derivative exists, the natural property that should be considered is the integral of the function. Taylor series and approximation formulas are useful in the computations when a parameter involved becomes large.

The analytic properties of the logarithmic Lambert function are presented in Section 2. Its applications to the adiabatic thermostatistics of the three-parameter entropy of classical ideal gas are derived in Section 3. Relationships among the specific heats are derived in Section 4 with some important remarks given. Finally, a conclusion is presented in Section 5.

2. Logarithmic Generalization of the Lambert W Function

A generalization of the Lambert W function will be called the logarithmic Lambert function denoted by $W_{\mathscr{L}}(x)$. Its formal definition is given below, and fundamental properties of this function are proven.

Definition 1. For any real number x and constant B, the logarithmic Lambert function $W_{\mathscr{L}}(x)$ is defined to be the solution to the equation

$$y \ln (By)e^y = x. \tag{12}$$

Observe that y cannot be zero. Moreover, By must be positive. By Definition 1, $y = W_{\mathscr{L}}(x)$. The derivatives of $W_{\mathscr{L}}(x)$ with respect to x can be readily determined as the following theorem shows.

Theorem 2. *The derivative of the logarithmic Lambert function is given by*

$$\frac{dW_{\mathscr{L}}(x)}{dx} = \frac{e^{-W_{\mathscr{L}}(x)}}{[W_{\mathscr{L}}(x) + 1] \ln BW_{\mathscr{L}}(x) + 1}. \tag{13}$$

Proof. Taking the derivative of both sides of (12) gives

$$\ln (By)ye^y \frac{dy}{dx} + (\ln (By) + 1)e^y \frac{dy}{dx} = 1, \tag{14}$$

from which

$$\frac{dy}{dx} = \frac{1}{[y \ln (By) + \ln (By) + 1]e^y}. \tag{15}$$

With $y = W_{\mathscr{L}}(x)$, (15) reduces to (13).

The integral of the logarithmic Lambert function is given in the next theorem.

Theorem 3. *The integral of $W_{\mathscr{L}}(x)$ is*

$$\int W_{\mathscr{L}}(x)\, dx = e^{W_{\mathscr{L}}(x)} \left[1 + \left(W_{\mathscr{L}}^2(x) - W_{\mathscr{L}}(x) + 1 \right) \ln (W_{\mathscr{L}}(x)) \right]$$

$$- 2E_i(W_{\mathscr{L}}(x)) + C, \tag{16}$$

where $E_i(x)$ is the exponential integral given by

$$E_i(x) = \int \frac{e^x}{x}\, dx. \tag{17}$$

Proof. From (12),

$$dx = (y \ln (By) + \ln (By) + 1)e^y dy. \tag{18}$$

Thus,

$$\int y dx = \int y(y \ln (By) + \ln (By) + 1)e^y dy$$
$$= \int y^2 e^y \ln (By) dy + \int y e^y \ln (By) dy + \int y e^y dy. \tag{19}$$

These integrals can be computed using integration by parts to obtain

$$\int y e^y dy = (y - 1)e^y + C_1, \tag{20}$$

$$\int y e^y \ln (By) dy = e^y((y - 1) \ln (By) - 1) + E_i(y) + C_2, \tag{21}$$

$$\int y^2 e^y \ln (By) dy = e^y \left[(y^2 - 2y + 2) \ln (By) - y + 3 \right] - 2E_i(y) + C_3, \tag{22}$$

where C_1, C_2, C_3 are constants. Substitution of (20), (22), and (21) to (19) with $C = C_1 + C_2 + C_3$ and writing $W_{\mathscr{L}}(x)$ for y will give (16).

The next theorem contains the Taylor series expansion of $W_{\mathscr{L}}(x)$.

Theorem 4. *Few terms of the Taylor series of $W_{\mathscr{L}}(x)$ about 0 are given below:*

$$W_{\mathscr{L}}(x) = \frac{1}{B} + e^{-(1/B)} x + \frac{2 + B}{2!} e^{-(2/B)} x^2 + \frac{\left(4B^2 + 9B + 9 \right)}{3!} e^{-(B/3)} x^3 + \cdots. \tag{23}$$

Proof. Being the inverse of the function defined by $x = y \ln (By)e^y$, the Lagrange inversion theorem is the key to obtain the Taylor series of the function $W_{\mathscr{L}}(x)$.

Let $f(y) = y \ln (By)e^y$. The function f is analytic for $By > 0$. Moreover, $f'(y) = [(y + 1) \ln (By) + 1]e^y, f'(1/B) = e^{1/B} \neq 0$, and for finite B, $f(1/B) = 0$. By the Lagrange inversion theorem (taking $a = 1/B$),

$$W_{\mathscr{L}}(x) = \frac{1}{B} + \sum_{n=1}^{\infty} g_n \frac{x^n}{n!}, \tag{24}$$

where

$$g_n = \lim_{y \to \frac{1}{B}} \frac{d^{n-1}}{dy^{n-1}} \left(\frac{y - 1/B}{f(y)} \right)^n. \tag{25}$$

The values of g_n for $n = 1, 2, 3$ are

$$\begin{aligned} g_1 &= e^{-(1/B)}, \\ g_2 &= (2 + B)e^{-(2/B)}, \\ g_3 &= \left(4B^2 + 9B + 9\right)e^{-(3/B)}. \end{aligned} \tag{26}$$

Substituting these values to (24) will yield (23).

An approximation formula for $W_{\mathscr{L}}(x)$ expressed in terms of the classical Lambert W function is proven in the next theorem.

Theorem 5. *For large x,*

$$W_{\mathscr{L}}(x) \sim W(x) - \ln\left(\ln\left(BW(x)\right)\right), \tag{27}$$

where $W(x)$ denotes the Lambert W function.

Proof. From (12), $y = W_{\mathscr{L}}(x)$ satisfies

$$x = y(\ln By)e^y \sim ye^y. \tag{28}$$

Then,

$$y = W(x) + u(x), \tag{29}$$

where $u(x)$ is a function to be determined. Substituting (29) to (12) yields

$$W(x)\left[1 + \frac{u(x)}{W(x)}\right] \ln\left(BW(x)\left[1 + \frac{u(x)}{W(x)}\right]\right) e^{W(x)} \cdot e^{u(x)} = x. \tag{30}$$

With $u(x) << W(x)$, (30) becomes

$$W(x)e^{W(x)} \ln\left(BW(x)\right) e^{u(x)} = x. \tag{31}$$

By definition of $W(x)$, $W(x)e^{W(x)} = x$. Hence, (31) gives

$$\ln\left(BW(x)\right)e^{u(x)} = 1, \tag{32}$$

from which

$$u(x) = -\ln\left(\ln\left(BW(x)\right)\right). \tag{33}$$

Thus,

$$W_{\mathscr{L}}(x) \sim W(x) - \ln\left(\ln\left(BW(x)\right)\right). \tag{34}$$

Table 2 illustrates the accuracy of the approximation formula in (27).

The next theorem describes the branches of the logarithmic Lambert function.

Theorem 6. *Let $x = f(y) = y \ln (By)e^y$. Then, the branches of the logarithmic Lambert function $y = W_{\mathscr{L}}(x)$ can be described as follows:*

(1) When $B > 0$, the branches are as follows:

  (i) $W_{\mathscr{L}}^0(x)$: $(f(\delta), +\infty) \longrightarrow [\delta, +\infty)$ is strictly increasing

  (ii) $W_{\mathscr{L}}^1(x)$: $(f(\delta), 0) \longrightarrow [0, \delta)$ is strictly decreasing

where δ is the unique solution to

$$(y + 1) \ln (By) = -1. \tag{35}$$

(2) When $B < 0$, the branches are as follows:

  (i) $W_{\mathscr{L},<}^0(x)$: $[0, f(\delta_2)] \longrightarrow [\delta_2, +\infty)$ is strictly decreasing

  (ii) $W_{\mathscr{L},<}^1(x)$: $[f(\delta_1), f(\delta_2)] \longrightarrow [\delta_1, \delta_2]$ is strictly increasing

  (iii) $W_{\mathscr{L},<}^2(x)$: $[f(\delta_1), 0] \longrightarrow (-\infty, \delta_1)$ is strictly decreasing

where δ_1 and δ_2 are the two solutions to (35) with $\delta_1 < 1/B < \delta_2 < 0$.

Proof. Consider the case when $B > 0$. Let $x = f(y) = y \ln (By)e^y$. From equation (13), the derivative of $y = W_{\mathscr{L}}(x)$ is not defined when y satisfies (35). The solution $y = \delta$ (35) can be viewed as the intersection of the functions:

$$\begin{aligned} g(y) &= -1, \\ h(y) &= (y + 1) \ln (By). \end{aligned} \tag{36}$$

Clearly, the solution is unique. Thus, the derivative $dW_{\mathscr{L}}(x)/dx$ is not defined for $x = f(\delta) = \delta \ln (B\delta)e^\delta$. The value of $f(\delta)$ can then be used to determine the branches of $W_{\mathscr{L}}(x)$. To explicitly identify the said branches, the following information is important:

(1) The value of y must always be positive; otherwise, $\ln (By)$ is undefined

(2) The function $y = W_{\mathscr{L}}(x)$ has only one y-intercept, i.e., $y = 1/B$

(3) If $y < \delta$, $(y + 1) \ln (By) + 1 < 0$ which gives $dy/dx < 0$

(4) If $y > \delta$, $(y + 1) \ln (By) + 1 > 0$ which gives $dy/dx > 0$

(5) If $y = \delta$, $(y + 1) \ln (By) + 1 = 0$ and dy/dx does not exist

Ensemble	Heat function	Entropy
Microcanonical (N, V, E)	Internal energy E	$S(N, V, E)$
Isoenthalpic-isobaric (N, P, H)	Enthalpy $H = E + PV$	$S(N, P, H)$
Third adiabatic ensemble (μ, V, L)	Hill energy $L = E - \mu N$	$S(\mu, V, L)$
Fourth adiabatic ensemble (μ, P, R)	Ray energy $R = E + PV - \mu N$	$S(\mu, P, R)$

TABLE 2

x	$W_{\mathscr{L}}(x)$	Approximate value	Relative error
302.7564	4	3.8914	2.71438×10^{-2}
1194.3088	5	4.8766	2.46807×10^{-2}
4337.0842	6	5.8756	2.07321×10^{-2}
14937.6471	7	6.8792	1.72518×10^{-2}
49589.8229	8	7.8844	1.44500×10^{-2}
160238.6564	9	8.8899	1.22306×10^{-2}
507178.1179	10	9.8953	1.04662×10^{-2}

These imply that

(1) when $y > \delta$, the function $y = W_{\mathscr{L}}(x)$ is increasing in the domain $(f(\delta), +\infty)$ with range $[\delta, +\infty)$ and the function crosses the y-axis only at $y = 1/B$

(2) when $y < \delta$, the function $y = W_{\mathscr{L}}(x)$ is decreasing, the domain is $(f(\delta), 0)$, and the range is $[\delta, 0)$ because this part of the graph does not cross the x-axis and y-axis

(3) when $y = \delta$, the line tangent to the curve at the point $(f(\delta), \delta)$ is a vertical line

These proved the case when $B > 0$. For the case $B < 0$, the solution to (35) can be viewed as the intersection of the functions:

$$g(y) = -\frac{1}{y+1},$$
$$h(y) = \ln(By). \tag{37}$$

These graphs intersect at two points δ_1 and δ_2. Thus, the derivative $dW_{\mathscr{L}}(x)/dx$ is not defined for

$$x_1 = f(\delta_1) = \delta_1 \ln(B\delta_1) e^{\delta_1},$$
$$x_2 = f(\delta_2) = \delta_2 \ln(B\delta_2) e^{\delta_2}. \tag{38}$$

Note that

(1) the value of y must always be negative; otherwise, ln (By) is undefined

(2) the function $y = W_{\mathscr{L}}(x)$ has only one y-intercept, i.e., $y = 1/B$

(3) $g(y)$ is not defined at $y = -1$

The desired branches are completely determined as follows:

(1) If $\delta_2 < y < 0$, then $(y + 1) \ln By + 1 < 0$. This gives $dy/dx < 0$. Thus, the function $y = W_{\mathscr{L}}(x)$ is a decreasing function with domain $[0, f(\delta_2)]$ and range $[\delta_2, 0]$

(2) If $\delta_1 \leq y \leq \delta_2$, then $(y + 1) \ln By + 1 > 0$. This gives $dy/dx > 0$. Thus, the function $y = W_{\mathscr{L}}(x)$ is an increasing function with domain $[f(\delta_1), f(\delta_2)]$ and range $[\delta_1, \delta_2]$

(3) If $-\infty < y < \delta_1$, then $(y + 1) \ln By + 1 < 0$. This gives $dy/dx < 0$. Thus, $y = W_{\mathscr{L}}(x)$ is a decreasing function with domain $[f(\delta_1), 0]$ and range $(-\infty, \delta_1]$

These complete the proof of the theorem.

Figure 1 depicts the graphs of the logarithmic Lambert function (red color) when $B = 1$ and $B = -1$. The y-coordinates of the points of intersection of the blue- and gray-colored graphs correspond to the value of $\delta, \delta_1, \delta_2$.

3. Applications to Classical Ideal Gas

As discussed in [3], the microstate of a system of N particles can be represented by a single point in the $2DN$-dimensional phase space. Corresponding to a particular value of the heat function which is a macrostate is a huge number of microstates. The total number of microstates has to be computed as it is a measure of entropy S. The points denoting the microstates of the system lie so close to each other that the surface area of the constant heat function curve in the phase space is regarded as a measure of the total number of microstates.

In an adiabatic ensemble, theoretically the density of states in the region E and $E + \Delta E$ should be calculated, where ΔE is very small. But it is not possible to calculate the number of states in this region. So the volume density of the states is calculated. Since the number of particles is very large, most of the volume of the region enclosed by a constant energy curve lies in its surface which is the surface density of states. So the volume density of states is equal to the surface density of states. Hence, in the calculation, the volume density of states is used.

In this section, applications of the logarithmic Lambert function to classical ideal gas in the four adiabatic ensembles
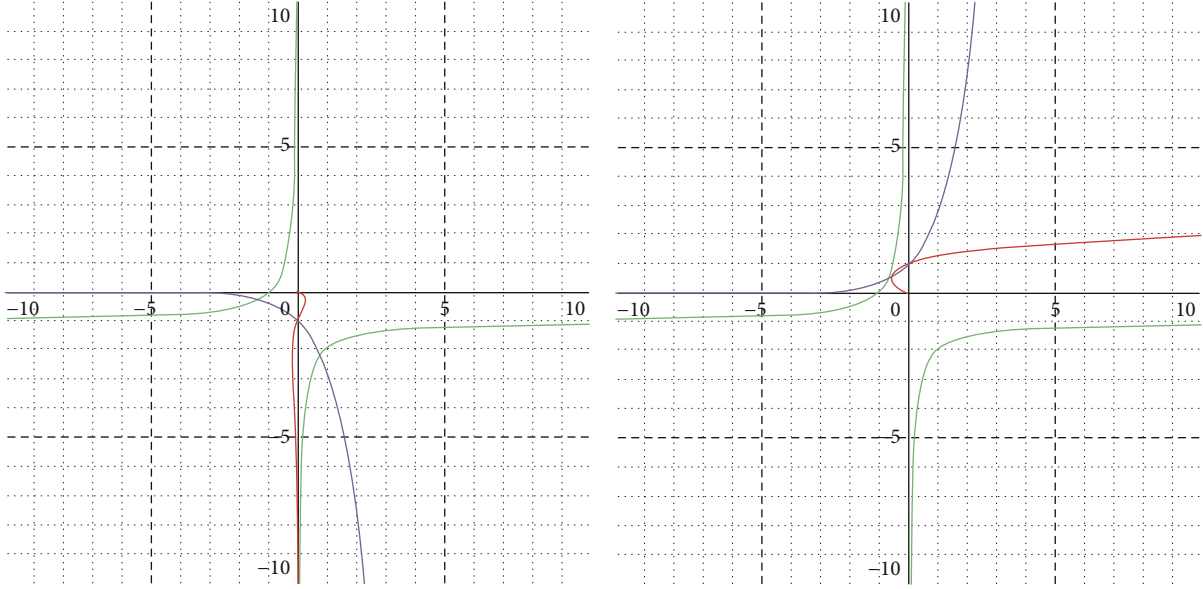
FIGURE 1: Graphs of the logarithmic Lambert function with $B = 1$ and $B = -1$. The graphs with red, blue, and gray colors are the graphs of $x = f(y)$, $x = g(y)$, and $x = h(y)$, respectively.

are derived. In what follows, m denotes the mass of the system; P, pressure; V, volume; h, Planck's constant (see [20], p. 119); and μ, the chemical potential of the system (see [21]).

3.1. Microcanonical Ensemble (N, V, E).

The Hamiltonian of a nonrelativistic classical ideal gas in D dimensions is

$$H = \sum_i \frac{P_i^2}{2m}, \quad P_i = |p_i|, \tag{39}$$

where P_i (for $i = 1, 2, \cdots, N$) represent the D-dimensional momentum of the gas molecules. This classical nonrelativistic ideal gas is studied in the microcanonical ensemble. In order to compute the entropy of the system, the phase space volume enclosed by the constant energy curve is computed and is given by (see [3])

$$\sum(N, V, E) = \frac{V^N}{N!} \frac{M^N}{\Gamma((DN/2) + 1)} E^{DN/2}, \tag{40}$$

where

$$M = \left(\frac{2\pi m}{h^2}\right)^{D/2}. \tag{41}$$

The three-parameter entropy of the system is

$$\begin{aligned} S_{q,q',r} &= k \ln_{q,q',r} \sum(N, V, E) \\ &= \frac{k}{1-r} \left[\exp\left(\frac{1-r}{1-q'} \exp\left(\frac{1-q'}{1-q}\left(\left(\sum(N, V, E)\right)^{1-q} - 1\right)\right) - 1\right) - 1 \right]. \end{aligned} \tag{42}$$

Computing the inner exponential,

$$\begin{aligned} &\exp\left(\frac{1-q'}{1-q}\left(\left(\sum(N, V, E)\right)^{1-q} - 1\right)\right) \\ &= \exp\left(\frac{1-q'}{1-q}\left(\left(\frac{V^N M^N E^{DN/2}}{N!\Gamma((DN/2)+1)}\right)^{1-q} - 1\right)\right) \\ &= \exp\left(\frac{1-q'}{1-q}\left(\xi_{mc}^{1-q} \cdot E^{(DN/2)(1-q)} - 1\right)\right), \end{aligned} \tag{43}$$

where

$$\xi_{mc} = \frac{V^N M^N}{N!\Gamma((DN/2)+1)}. \tag{44}$$

Let

$$u = \frac{1-q'}{1-q} \xi_{mc}^{1-q} E^{(DN/2)(1-q)}. \tag{45}$$

Then,

$$S_{q,q',r} = \frac{k}{1-r}\left[\exp\left(\frac{1-r}{1-q'}e^u A\right) e^{-(1-r)/(1-q')} - 1\right], \tag{46}$$

where

$$A = e^{\frac{-(1-q')}{1-q}}. \tag{47}$$

With

$$z = e^u, \tag{48}$$

$$S_{q,q',r} = \frac{k}{1-r}\left[e^{\left((1-r)/(1-q')\right)zA}e^{-(1-r)/(1-q')} - 1\right]. \qquad (49)$$

From the definition of temperature,

$$T = \left(\frac{\partial S_{q,q',r}}{\partial E}\right)^{-1}. \qquad (50)$$

Hence,

$$\frac{1}{T} = \frac{\partial S_{q,q',r}}{\partial E} = \frac{k}{1-r}\left[e^{-(1-r)/(1-q')}e^{\left((1-r)/(1-q')\right)zA}\left(\frac{1-r}{1-q'}\right)A\frac{dz}{du}\cdot\frac{du}{dE}\right], \qquad (51)$$

where

$$dz = e^u = z,$$
$$\frac{du}{dE} = \left(1-q'\right)\xi_{mc}^{1-q}\frac{DN}{2}E^{(DN/2)(1-q)-1}. \qquad (52)$$

Thus,

$$\frac{1}{T} = ke^{-(1-r)/(1-q')}e^{\left((1-r)/(1-q')\right)zA}zA\xi_{mc}^{1-q}\frac{DN}{2}E^{(DN/2)(1-q)-1}. \qquad (53)$$

Substituting

$$\beta = \frac{1}{kT}, \qquad (54)$$

the preceding equation becomes

$$\frac{\beta e^{(1-r)/(1-q')}}{(DN/2)A} = e^{\left((1-r)/(1-q')\right)e^u A}e^u\xi_{mc}^{1-q}E^{(DN/2)(1-q)-1}. \qquad (55)$$

Let

$$y = \frac{1-r}{1-q'}Ae^u. \qquad (56)$$

Then,

$$\frac{1-q}{\xi_{mc}^{1-q}\left(1-q'\right)}\ln\left(\frac{y\left(1-q'\right)}{(1-r)A}\right) = E^{(DN/2)(1-q)}. \qquad (57)$$

For large N,

$$\frac{DN}{2}(1-q) - 1 \approx \frac{DN}{2}(1-q). \qquad (58)$$

Then, (55) is approximated by

$$\frac{2\beta e^{(1-r)/(1-q')}(1-r)}{(1-q)DN} = ye^y\ln(By), \qquad (59)$$

where

$$B = \frac{1-q'}{(1-r)A}. \qquad (60)$$

Let

$$x = \frac{2\beta e^{(1-r)/(1-q')}(1-r)}{(1-q)DN}. \qquad (61)$$

It follows from (59) that

$$y = W_{\mathscr{L}}(x). \qquad (62)$$

From (57),

$$E^{(DN/2)(1-q)} = \frac{1-q}{\xi_{mc}^{1-q}\left(1-q'\right)}\ln\left(BW_{\mathscr{L}}\left(\frac{2\beta e^{(1-r)/(1-q')}(1-r)}{(1-q)DN}\right)\right), \qquad (63)$$

which gives

$$E = \left[\frac{1-q}{\xi_{mc}^{1-q}\left(1-q'\right)}\ln\left(BW_{\mathscr{L}}\left(\frac{2\beta e^{(1-r)/(1-q')}(1-r)}{(1-q)DN}\right)\right)\right]^{2/DN(1-q)}. \qquad (64)$$

The specific heat at constant volume is

$$C_V = \frac{\partial E}{\partial T}. \qquad (65)$$

Let

$$a = \frac{1-q}{\left(1-q'\right)\xi_{mc}^{1-q'}},$$
$$c = \frac{2(1-r)e^{(1-r)/(1-q')}}{(1-q)DN}. \qquad (66)$$

Then, (64) can be written

$$E = \left[a\ln\left(BW_{\mathscr{L}}\left(\frac{c}{kT}\right)\right)\right]^{2/(1-q)DN}. \qquad (67)$$

Taking the partial derivative of E with respect to T,

$$C_V = \frac{\partial E}{\partial T} = \frac{2a^{2/DN\,(1-q)}[\ln BW_{\mathscr{L}}(c/kT)]^{(2/DN\,(1-q))-1}}{W_{\mathscr{L}}(c/kT)}\frac{d}{dt}W_{\mathscr{L}}\left(\frac{c}{kT}\right), \qquad (68)$$

where

$$\frac{d}{dT}W_{\mathscr{L}}\left(\frac{c}{kT}\right) = \frac{-\left(c/kT^2\right)e^{-W_{\mathscr{L}}(c/kT)}}{[W_{\mathscr{L}}(c/kT) + 1]\ln BW_{\mathscr{L}}(c/kT) + 1}. \qquad (69)$$

Let us consider the following regions depending on the values of the deformation parameters q, q', and r. (i) When $r > 1$ and $q > 1$, the argument of $W_{\mathscr{L}}$ is positive. If $q' > 1$, then $B > 0$ and $W_{\mathscr{L}}$ must be the principal branch $W_{\mathscr{L}}^{0}$. If $q' < 1$, then $B < 0$. With the argument of $W_{\mathscr{L}}$ being positive, we shall take $W_{\mathscr{L}}$ to be the branch $W_{\mathscr{L},<}^{0}$. (ii) If $r < 1$ and $q < 1$, then the argument of $W_{\mathscr{L}}$ is positive. If $q' < 1$, then $B > 0$ and we take $W_{\mathscr{L}}$ to be the principal branch $W_{\mathscr{L}}^{0}$. If $q' > 1$, then $B < 0$ and again we take $W_{\mathscr{L}}$ to be the branch $W_{\mathscr{L},<}^{0}$. (iii) If $r < 1, q > 1$, then the argument of $W_{\mathscr{L}}$ is negative. If $q' < 1$, then B is positive and we take $W_{\mathscr{L}}$ to be the principal branch $W_{\mathscr{L}}^{0}$. If $q' > 1$, then B is negative. Here, we have two choices for $W_{\mathscr{L}}$, either $W_{\mathscr{L},<}^{1}$ or $W_{\mathscr{L},<}^{2}$. Since the heat function must be a continuous function of the deformation parameters, we must in this case restrict $W_{\mathscr{L}}$ to $W_{\mathscr{L},<}^{1}$.

The specific heat at constant pressure is either positive or negative depending on the values of the deformation parameters q, q' and r.

The molar heat capacity must be positive in a stable system [22]. In order to find the molar heat capacity of a compound or element, simply multiply the specific heat by the molar mass [23]. For the molar mass to be positive, the specific heat must be positive. Thus, stability of the system requires that specific heat is positive. Looking at (69), stability is guaranteed if $BW_{\mathscr{L}}(c/kT) > 1$ with B and $W_{\mathscr{L}}(c/kT)$ being dependent on the deformation parameters. The analysis above which is based on the deformation parameters will help determine the stability of the system. On the other hand, negative specific heat appears in astrophysical physics [24].

For some values of the parameters, the specific heat C_V is depicted in Figures 2(a) and 2(b).

In the applications to the three other adiabatic ensembles, the definition of M, A, z, β, y, and B will be the same as those in (41), (47), (48), (54), (56), and (60), respectively. However, since definition of u differs in every ensemble, z will have different values in every ensemble.

3.2. The Isoenthalpic-Isobaric Ensemble (N, P, H). A system which exchanges energy and volume with its surroundings in such a way that its enthalpy remains constant is described by the isoenthalpic-isobaric ensemble. In order to calculate the thermodynamic quantities, the phase space volume enclosed by the constant enthalpy curve was computed in [25] and was given by

$$\sum(N, P, H) = \frac{M^N}{N!} \frac{1}{\Gamma(((D+N)/2)+1)} \sum_V V^N (H-PV)^{DN/2}. \tag{70}$$

Since the volume states are very closely spread, the summation is replaced by an integration. But an integral over the volume leads to overcounting of the eigenstates. To overcome this, the shell particle method of counting volume states was employed (see [2, 15]). In this method, only the minimum volume needed to confine in a particular configuration is taken into account. The minimum volume needed to

confine a particular configuration is found by imposing a condition that requires at least one particle to lie on the boundary of the system. All the equivalent ways of choosing a minimum volume for a particular configuration are treated as the volume eigenstate and are considered only once. Using this shell particle technique to reject the redundant volume states, the following expression for the phase space volume was obtained (see [15]):

$$\sum(N, P, H) = M^N \left(\frac{1}{P}\right)^N \frac{H^{(DN/2)+N}}{\Gamma((DN/2)+N+1)}. \tag{71}$$

The three-parameter entropy of the system is

$$S_{q,q',r} = k \ln_{q,q',r} \sum(N, P, H) = \frac{k}{1-r}\left[e^{\left((1-r)/\left(1-q'\right)\right)zA}e^{-\left((1-r)/\left(1-q'\right)\right)} - 1\right], \tag{72}$$

where $\alpha = (DN/2) + N$,

$$u = \frac{1-q'}{1-q}\xi_{ie}^{1-q}H^{\alpha(1-q)}, \tag{73}$$

$$\xi_{ie} = \frac{M^N (1/P)^N}{\Gamma((DN/2)+N+1)}.$$

From the definition of temperature,

$$\frac{1}{T} = \frac{\partial S_{q,q',r}}{\partial H} = k\left[e^{-\left((1-r)/\left((1-q')\right)\right)}e^{\left((1-r)/\left(1-q'\right)\right)zA} \cdot \left(1-q'\right)A\frac{dz}{du}\frac{du}{dH}\right], \tag{74}$$

where

$$\frac{dz}{du} = e^u = z, \tag{75}$$

$$\frac{du}{dH} = \alpha\left(1-q'\right)\xi_{ie}^{1-q}H^{\alpha(1-q)-1}.$$

For large N, approximately

$$H^{\alpha(1-q)-1} \approx H^{\alpha(1-q)}. \tag{76}$$

Then,

$$\frac{du}{dH} \approx \alpha\left(1-q'\right)\xi_{ie}^{1-q}H^{\alpha(1-q)}, \tag{77}$$

$$\frac{1}{T} = k\left[e^{-\left((1-r)/\left(1-q'\right)\right)}e^{\left((1-r)/(1-q)'\right)zA}Az\alpha\,\xi_{ie}^{1-q}H^{\alpha(1-q)}\right]. \tag{78}$$

Substituting $\beta = 1/kT$, (78) becomes

$$\frac{\beta e^{(1-r)/\left(1-q'\right)}}{A\alpha\xi_{ie}^{1-q}} = e^{\left((1-r)/\left(1-q'\right)\right)zA}zH^{\alpha(1-q)}, \tag{79}$$
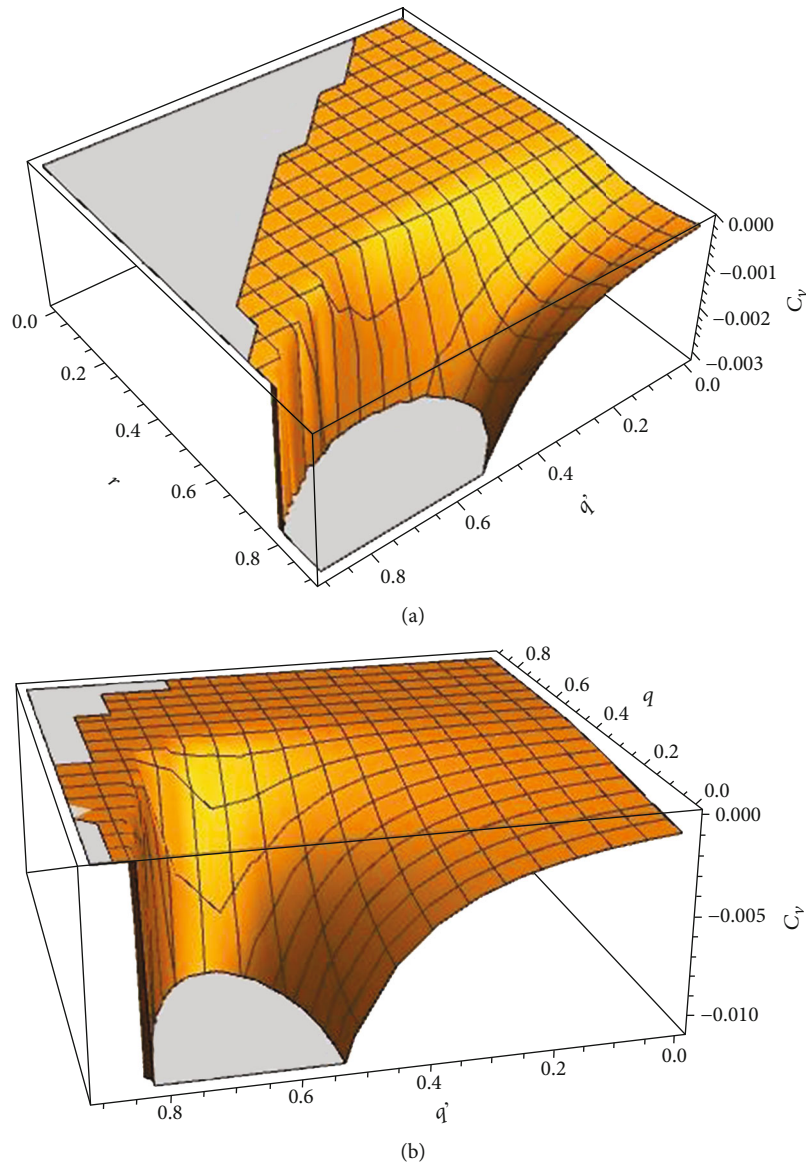
(a)

(b)

Figure 2: (a) Graph of C_V when q', r vary and $q = 0.4$, $T = 2$, $V = 3$, $m = 2$, and $h = D = N = k = A = 1$. (b) Graph of C_V when q, q' vary and $r = 0.6$, $T = 3$, $V = 4$, $m = 2$, $h = 1$, $D = 1$, and $N = k = A = 1$.

from which

$$H^{\alpha(1-q)} = \frac{1-q}{\xi_{ie}^{1-q}\left(1-q'\right)} \ln\left(\frac{y\left(1-q'\right)}{(1-r)A}\right), \qquad (80)$$

where y is defined in (56). Then, (79) can be written

$$\frac{\beta e^{(1-r)/\left(1-q'\right)}(1-r)}{\alpha(1-q)} = ye^{y} \ln (By). \qquad (81)$$
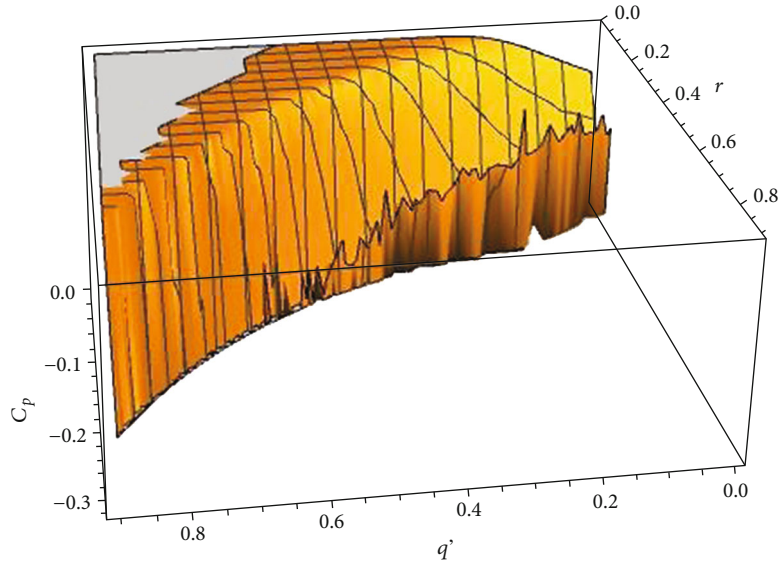
It follows from (81) that

$$y = W_{\mathscr{L}}\left(\frac{\beta e^{(1-r)/\left(1-q'\right)}(1-r)}{\alpha(1-q)}\right). \qquad (82)$$
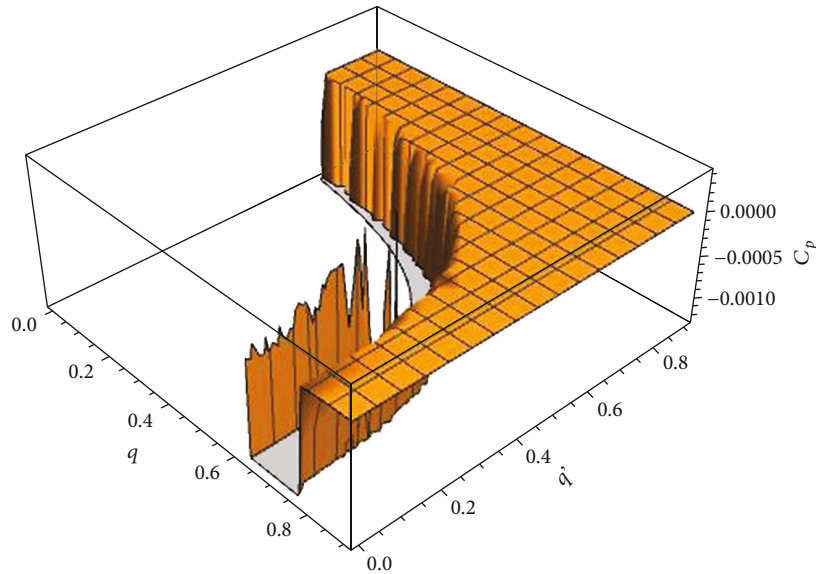
From (80),

$$H = \left[\frac{1-q}{\xi_{ie}^{1-q}\left(1-q'\right)} \ln\left(\frac{1-q'}{(1-r)A} W_{\mathscr{L}}\left(\frac{\beta e^{(1-r)/\left(1-q'\right)}(1-r)}{\alpha(1-q)}\right)\right)\right]^{1/\alpha(1-q)}. \qquad (83)$$

Let

$$a = \frac{1-q}{\xi_{ie}^{1-q}\left(1-q'\right)},$$

$$c = \frac{e^{(1-r)/\left(1-q'\right)}(1-r)}{\alpha(1-q)}. \qquad (84)$$

(a)

(b)

Figure 3: (a) Graph of C_P when q', r vary and $q = 0.4$, $T = 2$, $P = 3$, $M = 2$, $D = 1$, and $N = k = A = 1$. (b) Graph of C_P when q, q' vary and r $= 0.2$, $T = 3$, $P = 4$, $M = 1$, $D = 1$, and $N = k = A = 1$.

Then,

$$
H = \left[a \ln \left(BW_{\mathscr{L}} \left(\frac{c}{kT} \right) \right) \right]^{1/\alpha(1-q)}. \tag{85}
$$

The specific heat at constant pressure is

$$
C_P = \frac{\partial H}{\partial T}. \tag{86}
$$

Taking the partial derivative of (85),

$$
\frac{\partial H}{\partial T} = \frac{1}{\alpha(1-q)} \left[a \ln \left(BW_{\mathscr{L}} \left(\frac{c}{kT} \right) \right) \right]^{(1/\alpha(1-q))-1} \cdot \frac{a(d/dt) W_{\mathscr{L}}(c/kT)}{(W_{\mathscr{L}}(c/kT))}. \tag{87}
$$

Using the derivative of $W_{\mathscr{L}}(x)$ with respect to x given in Section 2,

$$
C_P = \frac{\left(-c/\alpha(1-q)kT^2 \right) a^{1/\alpha(1-q)} e^{-W_{\mathscr{L}}(c/kT)} \left[\ln \left(BW_{\mathscr{L}}(c/kT) \right) \right]^{(1/\alpha(1-q))-1}}{W_{\mathscr{L}}(c/kT) \{ [W_{\mathscr{L}}(c/kT) + 1] \ln BW_{\mathscr{L}}(c/kT) + 1 \}}. \tag{88}
$$

The specific heat at constant pressure is either positive or negative depending on the values of the deformation parameters q, q' and r. For some values of the parameters, the specific heat C_P is depicted in Figures 3(a) and 3(b).

3.3. The (μ, V, L) Ensemble.

The Hill energy L is the heat function corresponding to the (μ, V, L) ensemble. For large N, an approximate expression of the phase space volume obtained in [3] is

$$\sum(\mu, V, L) = \exp\left(\frac{D}{2}\frac{L}{\mu}\right) \exp\left(\frac{V\mu^{D/2}e^{D/2}M}{(D/2)^{D/2}}\right). \quad (89)$$

This is a first-order approximation of the exact value:

$$\sum(\mu, V, L) = \sum_{N=0}^{\infty} \frac{V^N}{N!} \frac{M^N}{\Gamma((DN/2)+1)}(L+\mu N)^{DN/2}. \quad (90)$$

The 3-parameter entropy of the classical ideal gas in this adiabatic ensemble is

$$S_{q,q',r} = \frac{k}{1-r}\left[\exp\left(\frac{1-r}{1-q'}zA\right)e^{-((1-r)/1-q')} - 1\right], \quad (91)$$

where $z = e^u$,

$$\xi_{he} = \exp\left(\frac{V(\mu e)^{D/2}M}{(D/2)^{D/2}}\right),$$
$$u = \frac{1-q'}{1-q}\xi_{he}^{1-q}e^{(DL/2\mu)(1-q)}. \quad (92)$$

From the definition of temperature,

$$\frac{1}{T} = \frac{\partial S_{q,q',r}}{\partial H} = k\left[e^{-((1-r)/(1-q'))}e^{((1-r)/(1-q'))zA} \cdot \frac{A}{1-q'}A\frac{dz}{du}\frac{du}{dL}\right], \quad (93)$$

where

$$\frac{du}{dL} = \frac{(1-q')D}{2\mu}\xi_{he}^{1-q}e^{(DL/2\mu)(1-q)}. \quad (94)$$

Then, (93) becomes

$$\frac{2\beta\mu\exp\left((1-r)/(1-q')\right)}{AD\xi_{he}^{1-q}} = \exp\left(\frac{1-r}{1-q'}zA\right)ze^{(DL/2\mu)(1-q)}, \quad (95)$$

and further

$$\frac{1-q}{(1-q')\xi_{ie}^{1-q}}\ln\left(\frac{y(1-q')}{(1-r)A}\right) = e^{(DL/2\mu)(1-q)}, \quad (96)$$

from which

$$L = \frac{2\mu}{D(1-q)}\ln\left(\frac{1-q}{(1-q')\xi_{ie}^{1-q}}\ln\left(\frac{y(1-q')}{(1-r)A}\right)\right). \quad (97)$$

To solve for y, rewrite (95) into

$$\frac{\beta(1-r)e^{(1-r)/(1-q')}}{(D/2\mu)(1-q)} = ye^y\ln\left(\frac{y(1-q')}{(1-r)A}\right). \quad (98)$$

Thus,

$$y = W_{\mathscr{L}}\left(\frac{2\beta\mu e^{(1-r)/(1-q')}(1-r)}{D(1-q)}\right). \quad (99)$$

Therefore,

$$L = \frac{2\mu}{D(1-q)}\ln\left(\frac{1-q}{(1-q')\xi_{he}^{1-q}}\ln\left(BW_{\mathscr{L}}\left(\frac{2\beta\mu e^{(1-r)/(1-q')}(1-r)}{D(1-q)}\right)\right)\right). \quad (100)$$

The specific heat at constant volume is

$$C_{V*} = \frac{\partial L}{\partial T}. \quad (101)$$

Let

$$a = \frac{1-q}{(1-q')\xi_{ie}^{1-q}},$$
$$c = \frac{2\mu(1-r)e^{(1-r)/(1-q')}}{D(1-q)}. \quad (102)$$

Then,

$$L = \frac{2\mu}{D(1-q)}\ln\left(a\ln\left(BW_{\mathscr{L}}(c\beta)\right)\right),$$
$$\frac{\partial L}{\partial T} = \frac{2\mu}{D(1-q)} \cdot \frac{(d/dt)\ln\left(BW_{\mathscr{L}}(c\beta)\right)}{\ln\left(BW_{\mathscr{L}}(c\beta)\right)}, \quad (103)$$

where

$$\frac{d}{dT}\ln\left(BW_{\mathscr{L}}(c\beta)\right) = -\frac{(c/kT^2)e^{-W_{\mathscr{L}}(c/kT)}}{W_{\mathscr{L}}(c/kT)}\left\{\left[W_{\mathscr{L}}\left(\frac{c}{kT}\right)+1\right]\ln\left(BW_{\mathscr{L}}\left(\frac{c}{kT}\right)\right)+1\right\}. \quad (104)$$

Thus,

$$C_{V*} = \frac{2\mu}{D(1-q)}\frac{\left(-(c/kT^2)\right)e^{-W_{\mathscr{L}}(c/kT)}\left[\ln\left(BW_{\mathscr{L}}(c/kT)\right)\right]^{-1}}{W_{\mathscr{L}}(c/kT)([W_{\mathscr{L}}(c/kT)+1]\ln\left(BW_{\mathscr{L}}(c/kT)\right)+1)}. \quad (105)$$

The specific heat at constant volume is either positive or negative depending on the values of the deformation parameters q, q' and r. For some values of the parameters, the specific heat C_{V*} is depicted in Figures 4(a) and 4(b).

3.4. The (μ, P, R) Ensemble. The adiabatic ensemble with both the number and volume fluctuations is illustrated here using the classical ideal gas.

In the large N limit, the expression of the phase space volume [3] is

$$\sum(\mu, P, R) = \exp\left(\frac{DR}{\mu}\right)\left[1 - \frac{M}{P}\left(\frac{\mu}{\alpha}\right)^{\alpha}\exp\alpha\right]^{-1}, \quad (106)$$

where $\alpha = (DN/2) + N$.

The three-parameter entropy is

$$S_{q,q',r} = \frac{k}{1-r}\left[e^{\left((1-r)/(1-q')\right)zA} \cdot e^{-\left((1-r)/(1-q')\right)} - 1\right], \quad (107)$$

where $z = e^u$,

$$\xi_{re} = \left(1 - \frac{M}{P}\left(\frac{\mu}{\alpha}\right)^{\alpha}e^{\alpha}\right)^{-1},$$
$$u = \frac{1-q'}{1-q}\xi_{re}^{1-q}e^{(DR/\mu)(1-q)}. \quad (108)$$

From the definition of temperature,

$$\frac{1}{T} = \frac{\partial S_{q,q',r}}{\partial R} = k\left[e^{-\left((1-r)/(1-q')\right)}e^{\left((1-r)/(1-q')\right)zA} \cdot \frac{A}{1-q'} \cdot \frac{dz}{du} \cdot \frac{du}{dR}\right], \quad (109)$$

where

$$\frac{du}{dR} = \frac{\left(1-q'\right)D}{\mu}\xi_{re}^{1-q}e^{(DR/\mu)(1-q)}. \quad (110)$$

Then,

$$\frac{1}{T} = k\left[e^{-\left((1-r)/(1-q')\right)}e^{\left((1-r)/(1-q')\right)zA}Az\frac{D}{\mu}\xi_{re}^{1-q}e^{(DR/\mu)(1-q)}\right], \quad (111)$$

from which

$$\frac{\mu\beta e^{(1-r)/(1-q')}}{AD\xi_{re}^{1-q}} = e^{\left((1-r)/(1-q')\right)zA}ze^{(DR/\mu)(1-q)}. \quad (112)$$

Then,

$$R = \frac{\mu}{D(1-q)}\ln\left(\frac{1-q}{\left(1-q'\right)\xi_{ie}^{1-q}}\ln\left(\frac{y\left(1-q'\right)}{(1-r)A}\right)\right). \quad (113)$$

From (112),

$$\frac{\mu\beta(1-r)e^{(1-r)/(1-q')}}{D(1-q)} = ye^{y}\ln(By). \quad (114)$$

Thus,

$$y = W_{\mathscr{L}}\left(\frac{\mu\beta(1-r)e^{(1-r)/(1-q')}}{D(1-q)}\right). \quad (115)$$

Consequently, from (113), the Ray energy of the system is

$$R = \frac{\mu}{D(1-q)}\times\ln\left(\frac{1-q}{\left(1-q'\right)\xi_{re}^{1-q}}\ln\left(\frac{\left(1-q'\right)}{(1-r)A}W_{\mathscr{L}}\left(\frac{\mu\beta(1-r)e^{(1-r)/(1-q')}}{D(1-q)}\right)\right)\right). \quad (116)$$

The specific heat at constant pressure is

$$C_{P*} = \frac{\partial R}{\partial T}. \quad (117)$$

Let

$$a = \frac{1-q}{\left(1-q'\right)\xi_{Re}^{1-q}},$$
$$c = \frac{\mu(1-r)e^{(1-r)/(1-q')}}{D(1-q)}. \quad (118)$$

Then,

$$R = \frac{\mu}{D(1-q)}\ln\left(a\ln\left(BW_{\mathscr{L}}\left(\frac{c}{kT}\right)\right)\right), \quad (119)$$

$$C_{P*} = \frac{\mu}{D(1-q)}\times\frac{\left(-(c/kT^2)\right)e^{-W_{\mathscr{L}}(c/kT)}[\ln(BW_{\mathscr{L}}(c/kT))]^{-1}}{W_{\mathscr{L}}(c/kT)\{\ln(BW_{\mathscr{L}}(c/kT))W_{\mathscr{L}}(c/kT) + \ln(BW_{\mathscr{L}}(c/kT)) + 1\}}. \quad (120)$$

The specific heat at constant pressure is either positive or negative depending on the values of the deformation parameters q, q' and r. For some values of the parameters, the specific heat C_{P*} is depicted in Figures 5(a) and 5(b).
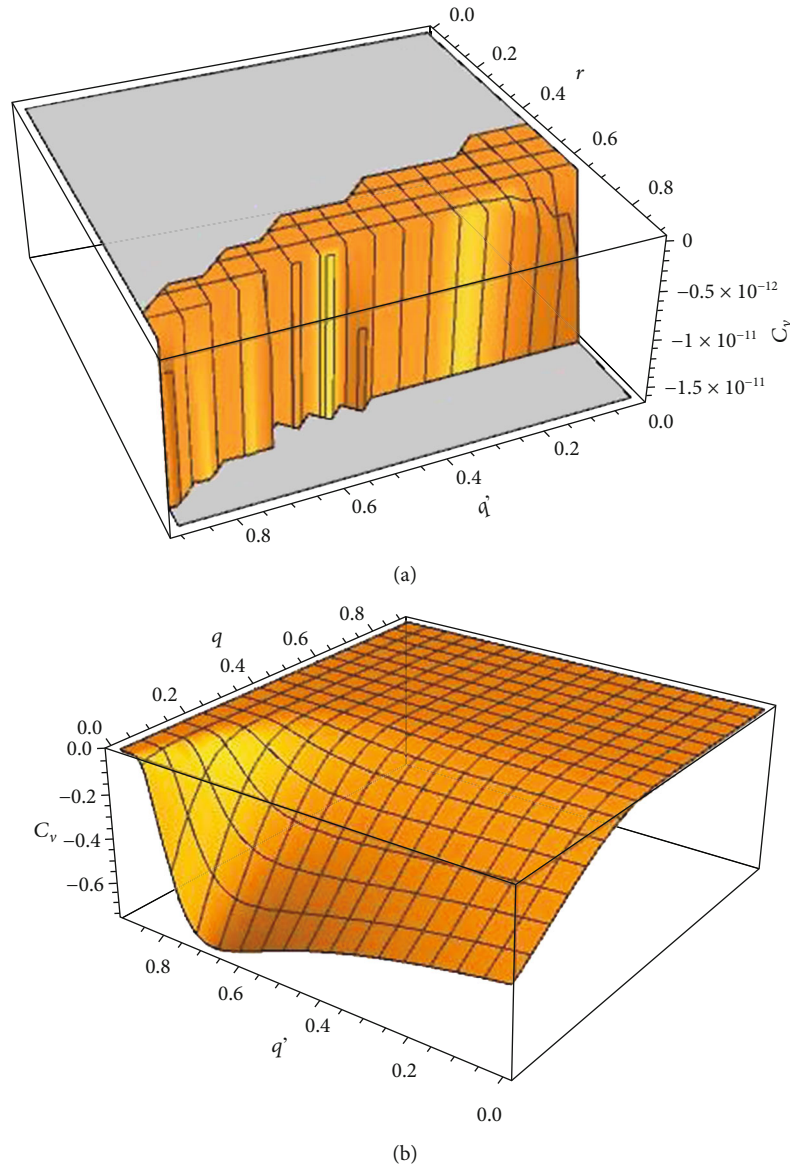
(a)

(b)

FIGURE 4: (a) Graph of C_{V*} when q', r vary and $q = 0.4, T = 2$, $P = 3$, $D = 1$, and $k = A = 1$. (b) Graph of C_{V*} when q, q' vary and $r = 0.8$, $T = 3$, $P = 4$, $D = 1$, and $k = A = 1$.

4. Relationship among the Specific Heats and Some Remarks

The relationship among the specific heats in the different ensembles is determined as follows. Writing

$$X = \frac{\left(-\left(c/kT^2\right)\right)e^{-W_{\mathscr{L}}(c/kT)}}{W_{\mathscr{L}}(c/kT)\{\ln BW_{\mathscr{L}}(c/kT)[W_{\mathscr{L}}(c/kT) + 1] + 1\}}, \quad (121)$$

then the specific heats can be written:

$$C_V = 2 X a^{(1/(\alpha-N))(1-q)} \frac{[\ln BW_{\mathscr{L}}(c/kT)]^{1/(\alpha-N)(1-q)}}{\ln BW_{\mathscr{L}}(c/kT)}, \quad (122)$$

$$C_P = \frac{X a^{1/\alpha(1-q)}}{\alpha(1-q)} \frac{[\ln BW_{\mathscr{L}}(c/kT)]^{1/\alpha(1-q)}}{\ln BW_{\mathscr{L}}(c/kT)}, \quad (123)$$

$$C_{V*} = 2 \frac{\mu X}{D(1-q) \ln BW_{\mathscr{L}}(c/kT)}, \quad (124)$$

$$C_{P*} = \frac{\mu X}{D(1-q) \ln BW_{\mathscr{L}}(c/kT)}. \quad (125)$$

From (122) and (123), we have the relation between the specific heats in the microcanonical ensemble and the
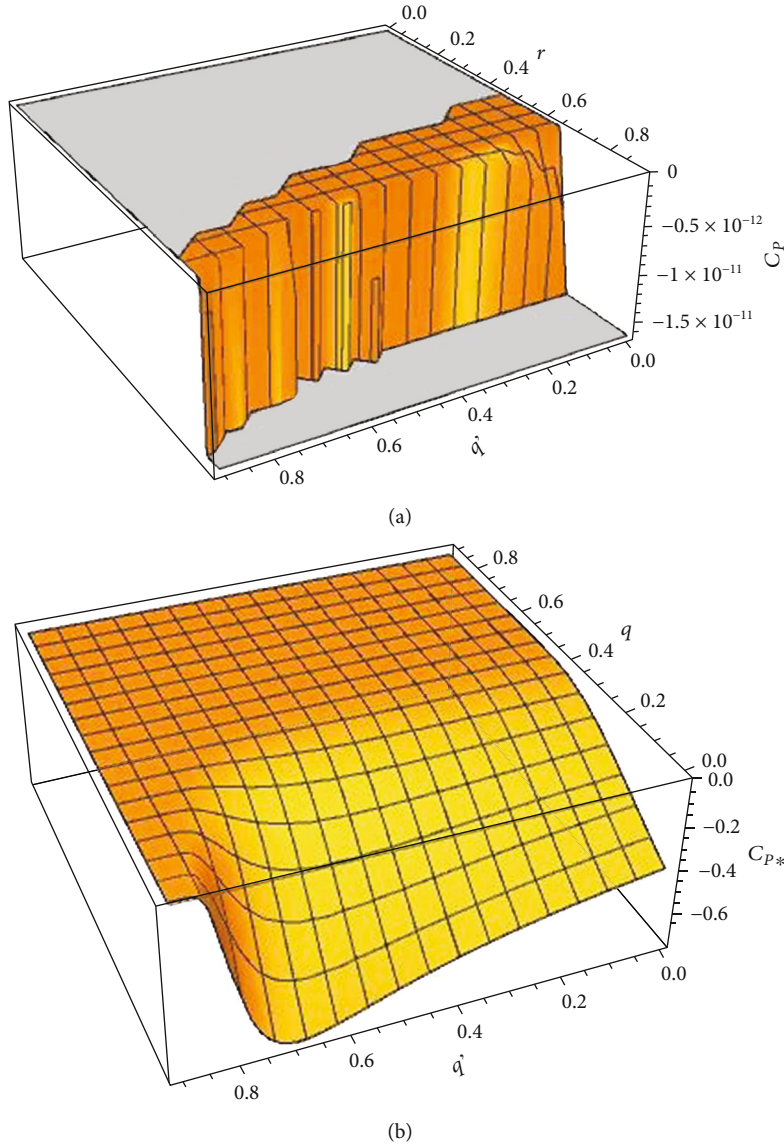
(a)

(b)

FIGURE 5: (a) Graph of C_{P_*} when q', r vary and $q = 0.4$, $T = 2$, $P = 3$, $D = 1$, and $k = A = 1$. (b) Graph of C_{P_*} when q, q' vary and $r = 0.8$, $T = 3$, $P = 4$, $D = 1$, and $k = A = 1$.

isoenthalpic-isobaric ensemble:

$$C_V = 2\left\{ \alpha(1-q)X^{-(N/\alpha)}C_P\left[\ln BW_{\mathscr{L}}\left(\frac{c}{kT}\right)\right]^{N/\alpha}\right\}^{1/(1-(N/\alpha))}.$$
$$(126)$$

From (124) and (125), we have the relation between the specific heats in the (μ, V, L) ensemble and the (μ, P, R) ensemble:

$$C_{V*} = 2C_{P*}.\qquad(127)$$

4.1. Some Remarks. As discussed in [19], if entropy is nonextensive, the physical temperature is not simply the inverse of the Lagrange multiplier associated with the energy constraint but a variable correctly defined through the generalized zeroth law of thermodynamics. The definition of the physical pressure also becomes different from the ordinary one. Taking into account these facts as well as the first law of thermodynamics and Legendre transform structure, it was shown in [19] that Clausius' definition of the thermodynamic entropy has to be appropriately modified.

In the above computation of the heat functions and specific heats, we used the inverse of the Lagrange multiplier β, in particular $T = 1/k\beta$, for the temperature of the system. As this is no longer the physical temperature for nonextensive entropy, we say that we used the "unphysical temperature," the term being used also to refer to the same expression in [19].

The results in [3] corresponding to (64) and (68), (85) and (88), (97) and (105), and (116) and (120) cannot be

recovered even when $r \longrightarrow 1$. The reason is that in the generalized statistical mechanics, the thermodynamic limit $N \longrightarrow \infty$ and the nonextensive limit $q \longrightarrow 1$ do not commute with each other [24, 26]. In the context of the translational specific heat, it was observed [24, 26] that the classical $q \longrightarrow 1$ limit and the thermodynamic limit do not commute. While equations (4.14) and (4.15) of [25] similarly demonstrate that the thermodynamic limit $N \longrightarrow \infty$ of the rotational and the diatomic specific heats does not commute with the corresponding $q \longrightarrow 1$ classical limits.

5. Conclusion

We have investigated the adiabatic class of ensembles in the framework of generalized mechanics based on the three-parameter entropy. The derivative and branches of the function were in particular useful in the applications to the thermostatistics of the nonrelativistic ideal gas. In the microcanonical ensemble and isoenthalpic-isobaric ensemble, the formulas for the three-parameter entropy for an arbitrary number of particles were obtained. In the large N limit, the heat functions were obtained in terms of the temperature and expressed in terms of the logarithmic Lambert function. From the heat functions, the specific heats at constant temperature were computed. In the (μ, V, L) and the (μ, P, R) ensembles, an approximate phase volume in the large N limit was used and the three-parameter entropies of the ensembles were computed. From the entropy function, the heat function and the specific heat were found and expressed also in terms of the logarithmic Lambert function.

Reexamination of the classical gas model using the appropriate expression for the physical temperature will be explored in a separate paper. The authors, motivated by the applications of the maximum entropy theory (MaxEnt) in ecology [27], also find it interesting to use the q-entropy, (q, q')-entropy, and (q, q', r)-entropy to formulate a generalized maximum entropy theory in ecology.

Data Availability

The computer programs and articles used to generate the graphs and support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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