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Research Article

Global Existence and Large Time Behavior for the 2-D Compressible Navier-Stokes Equations without Heat Conductivity

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In this paper, we consider an initial value problem for the 2-D compressible Navier-Stokes equations without heat conductivity. We prove the global existence of a strong solution when the initial perturbation is small in H^2 and its L^1 norm is bounded. Moreover, we derive some decay estimate for such a solution.

1. Introduction

The 2-D compressible Navier-Stokes equations for $(x, t) \in \mathbb{R}^2 \times \mathbb{R}^+$ are rewritten as

$$\begin{cases} \rho_t + \text{div}\,(\rho u) = 0, \\ (\rho u)_t + \text{div}\,(\rho u \otimes u) + \nabla P(\rho, \theta) = \text{div}\,\mathscr{T}, \\ (\rho \mathscr{E})_t + \text{div}\,(\rho \mathscr{E} u + P u) = \text{div}\,(u \mathscr{T}) + k \Delta \theta, \end{cases} \tag{1}$$

which govern the motion of gases, where ρ, u, P, θ stand for the density, velocity, pressure, and absolute temperature functions, respectively. $\mathscr{E} = (1/2)|u|^2 + E$ is the specific total energy with E as the specific internal energy, $\mathscr{T} = \mu(\nabla u + \nabla u^T) + \lambda(\text{div}\, u)I$ is the stress tensor, k is the coefficient of heat conduction, I is the identity matrix, and μ and λ are the coefficients of viscosity and second coefficient of viscosity, respectively, satisfying

$$\mu > 0, \lambda + \mu \geq 0. \tag{2}$$

As one of the most important systems in fluid dynamics, there are lots of results on the well-posedness, blow-up phenomenon, large time or asymptotic behavior, and optimal decay rates of solutions based on different assumptions in different cases and function spaces. Among them, for the case with a positive coefficient of heat conduction $k > 0$, Kazhikhov and Shelukhin studied the global existence in one dimension [1, 2]. The global existence of multidimensional case was established in [3–6]; more results on global existence for different kinds of solutions can be found in [3–10]. For the study of the large time behavior, asymptotic behavior, and optimal decay rates of solutions, one can refer to [4, 11–15]. The references [15–17] and [10, 18] restricted the systems under the case of $k = 0$ and $k > 0$, respectively. Danchin [8, 9] proved the existence and uniqueness of strong solutions to the compressible Navier-Stokes equations in hybrid Besov spaces, and Tan and Wang [6] studied the global existence of strong small solutions in $H^l, l \geq 4$. For the case of $k = 0$, Tan and Wang [5] proved the global solvability in three-dimensional space for the less regular solutions to the compressible Navier-Stokes equations in the H^2-framework; however, they needed to assume that the L^1-norm of the initial perturbation is bounded which is important in the proof of global existence. Later, ref. [3] removed this assumption by using some techniques with regard to the homogeneous Besov space and the hybrid Besov space.

Compared to the Cauchy problem, the equilibrium state of pressure increases with time. Xin et al. [16, 17, 19] investigated the blow-up phenomenon of the compressible Navier-Stokes equations in inhomogeneous Sobolev space; they proved that the smooth or strong solutions will blow up in any positive time if the initial data have an isolated mass group, no matter how small they are. On the other hand, we would like to introduce some research on the Serrin-type regularity (blow-up) criteria for the incompressible Navier-Stokes system. These criteria are obtained from [20, 21]; later, many authors successfully extended these blow-up criteria to the compressible flow (for example, [22–29] and references therein).

In this paper, we consider this problem in $\mathbb{R}^2$ with the case $k = 0$ and assume that the gas is ideal and polytropic, i.e., $P = R\rho\theta, E = c_v\theta, P = A e^{\mathcal{S}/c_v}\rho^\gamma$, where $R > 0$ and $A > 0$ are the universal gas constant, $\gamma > 1$ is the adiabatic exponent, $\mathcal{S}$ is the entropy, and $c_v = R/(\gamma - 1)$ is a constant which represents the specific heat at a constant volume. Furthermore, we also assume that $R = A = 1$ without loss of generality. Then, we have

$$\rho = P^{c_v/(c_v+1)} e^{-(\mathcal{S}/(c_v+1))}, \tag{3}$$

and system (1) in terms of variables $\rho, u, \mathcal{E}$ can be reformulated in terms of variables $P, u, \mathcal{S}$:

$$\begin{cases} P_t + \dfrac{1+c_v}{c_v} P \operatorname{div} u + u \cdot \nabla P = \dfrac{\Psi[u]}{c_v}, \\ u_t + (u\cdot\nabla)u + \dfrac{\nabla P}{\rho} = \dfrac{\mu}{\rho}\Delta u + \dfrac{\mu+\lambda}{\rho}\nabla(\nabla\cdot u), \\ \mathcal{S}_t + (u\cdot\nabla)\mathcal{S} = \dfrac{\Psi[u]}{P}, \end{cases} \tag{4}$$

where $\Psi[u] = (\mu/2)|\nabla u + \nabla u^T|^2 + \lambda(\operatorname{div} u)^2$ is the classical dissipation function. We are concerned with the initial value problem to system (4) with initial data satisfying

$$(P, u, \mathcal{S})(0, x) = (P_0, u_0, \mathcal{S}_0)(x) \longrightarrow (p_\infty, 0, s_\infty) \quad \text{as } |x| \longrightarrow \infty, \tag{5}$$

where $p_\infty > 0$ and s_∞ are the given constants. For the global existence and the decay estimate in the case of two dimensions, we have the following theorem.

Theorem 1. *Let $p_\infty > 0, s_\infty$ be two constants. There exists a small constant ε_0 such that if $(P_0 - p_\infty, u_0, \mathcal{S}_0 - s_\infty) \in H^2(\mathbb{R}^2)$, $\|P_0 - p_\infty, u_0, \mathcal{S}_0 - s_\infty\|_2 < \varepsilon_0$, and $\|P_0 - p_\infty, u_0\|_{L^1(\mathbb{R}^2)}$ are bounded, then there exists a unique global solution $(P, u, \mathcal{S})$ of the initial value problems (4) and (5) satisfying*

$$\begin{aligned} &\|(P - p_\infty, u)(t)\|_2^2 + \int_0^t \|\nabla P(\tau)\|_1^2 + \|\nabla u(\tau)\|_2^2 d\tau \le C\|(P_0 - p_\infty, u_0)\|_2^2, \\ &\|(\mathcal{S} - s_\infty)(t)\|_2 \le C\|(P_0 - p_\infty, u_0, \mathcal{S}_0 - s_\infty)\|_2 \exp\left(C\|(P_0 - p_\infty, u_0)\|_2\right). \end{aligned} \tag{6}$$

Finally, there is a constant C_0 such that for any $t \ge 0$, the solution $(P, u, \mathcal{S})$ has the decay properties

$$\begin{aligned} &\|\nabla(P - p_\infty, u)(t)\|_1 \le C_0(1+t)^{-1}, \\ &\|(P - p_\infty, u)(t)\|_{L^q} \le C_0(1+t)^{1/q-1}, \quad 2 \le q \le 4, \\ &\|(P - p_\infty, u)(t)\|_{L^q} \le C_0(1+t)^{-((2+q)/2q)}, \quad 4 < q \le \infty, \\ &\|\partial_t(P, u, \mathcal{S})(t)\|_{L^2} \le C_0(1+t)^{-(1/2)}. \end{aligned} \tag{7}$$

Remark 2.

(1) The L^q decay estimates of (7) for $2 \le q \le 4$ are optimal which coincide with the L^q decay of the heat equation and are much slower than the decay rate in $\mathbb{R}^3$. In [3, 5], they obtain that $\|u(t)\|_{L^2(\mathbb{R}^3)} \le C(1+t)^{-(3/4)}$, which ensures that $\int_0^\infty \|u(t)\|_{L^2(\mathbb{R}^3)}^2 dt$ is bounded. But in $\mathbb{R}^2$, we only have $\|u(t)\|_{L^2(\mathbb{R}^2)} \le C(1+t)^{-(1/2)}$, and therefore, $\int_0^\infty \|u(t)\|_{L^2(\mathbb{R}^2)}^2 dt$ is unbounded. This is the main difficulty about the proof of existence in $\mathbb{R}^2$ (see Section 3 below)

(2) Due to $k = 0$, we cannot gain any diffusion of $\mathcal{S}$, and the L^∞ decay estimate of (7) is slower than the decay of the heat equation

1.1. Notation. In this paper, we use L^p, H^m to denote the L^p and Sobolev spaces on $\mathbb{R}^2$ with norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{H^m} = \|\cdot\|_m$, respectively. We use C to denote the constants depending only on physical coefficients and C_0 to be constants depending additionally on the initial data.

This paper is organized as follows: In Section 2, we reformulate problems (4) and (5), introduce two main propositions, and illustrate that we only need to prove Proposition 4. The rest of the paper is devoted to proving Proposition 4. The proof of the energy estimate part is in Section 3, and the decay rate part is in Section 4.

2. Reformulated System

We reformulate system (4) by setting

$$\alpha_1 = \sqrt{\frac{c_v}{(1+c_v)\rho_\infty p_\infty}}, \quad \alpha_2 = \sqrt{\frac{(1+c_v)p_\infty}{c_v \rho_\infty}}, \quad \alpha_3 = \frac{\mu}{\rho_\infty}, \quad \alpha_4 = \frac{\mu+\lambda}{\rho_\infty}, \tag{8}$$

where $\rho_\infty = \rho(p_\infty, s_\infty)$.

Changing variables as $(P, u, \mathcal{S}) \longrightarrow (p + p_\infty, \alpha_1 v, s + s_\infty)$, initial value problems (4) and (5) are reformulated as

$$\begin{cases} p_t + \alpha_2 \nabla\cdot v = f, \\ v_t + \alpha_2 \nabla p - \alpha_3 \Delta v - \alpha_4 \nabla\nabla\cdot v = g, \\ s_t + \alpha_1 (v\cdot\nabla)s = h, \\ (p, v, s)(0, x) = (p_0, v_0, s_0)(x) \longrightarrow (0,0,0) \quad \text{as } |x| \longrightarrow \infty, \end{cases} \tag{9}$$

where the nonlinearities are given by

$$f(p,v,s) = -\frac{(1+c_v)\alpha_1}{c_v} p\nabla\cdot v - \alpha_1 v\cdot\nabla p + \frac{\Psi(\alpha_1 v)}{c_v},$$
$$g(p,v,s) = -\alpha_1 (v\cdot\nabla)v - \frac{1}{\alpha_1}\left(\frac{1}{\rho} - \frac{1}{\rho_\infty}\right)\nabla p + \mu\left(\frac{1}{\rho} - \frac{1}{\rho_\infty}\right)\Delta u + (\mu+\lambda)\left(\frac{1}{\rho} - \frac{1}{\rho_\infty}\right)\nabla\nabla\cdot u,$$
$$h(p,v,s) = \frac{\Psi(\alpha_1 v)}{p + p_\infty}. \quad (10)$$

For any $T > 0$, we define the solution space by

$$X(0,T) = \{(p,v,s): p, s\in C^0([0,T);H^2(\mathbb{R}^2))\cap C^1([0,T); \cdot H^1(\mathbb{R}^2)), v\in C^0([0,T);H^2(\mathbb{R}^2))\cap C^1([0,T); \cdot L^2(\mathbb{R}^2)), \nabla p\in L^2((0,T);H^1(\mathbb{R}^2)), \nabla v\in L^2((0,T);H^2(\mathbb{R}^2))\}, \quad (11)$$

and the solution norm by

$$\mathcal{X}(T) = \sup_{0\le t\le T}\|(p,v,s)(t)\|_2^2 + \int_0^T \|\nabla p(\tau)\|_1^2 + \|\nabla v(\tau)\|_2^2 d\tau. \quad (12)$$

Taking a standard contraction mapping argument, we have the following propositions for the local existence (see [30]).

Proposition 3 (see [3]). *Suppose that the initial data satisfy $(p_0, v_0, s_0)\in H^2(\mathbb{R}^2)$ and $\inf_{x\in\mathbb{R}^2}\{p_0 + p_\infty\} > 0$. Then, there exists a positive constant T_0 depending on $\mathcal{X}(0)$ such that Cauchy problem (9) has a unique solution $(p,v,s)\in X(0, T_0)$ satisfying*

$$\inf_{t\in[0,T_0],x\in\mathbb{R}^2}\{p(t,x) + p_\infty\} > 0, \mathcal{X}(T_0) \le 2\mathcal{X}(0). \quad (13)$$

This, together with the proposition below, is sufficient to derive Theorem 1; the proof is based on the standard continuity argument.

Proposition 4. *Let $(p_0, v_0, s_0)\in H^2(\mathbb{R}^2)$ and $(p_0, v_0)\in L^1(\mathbb{R}^2)$; for any $T > 0$, there exists $\varepsilon > 0$ such that the solution (p, v, s) of the initial value problem (9) in $[0, T]$ satisfies*

$$\|(p,v,s)\|_2 < \varepsilon, \quad \forall t\in[0,T]. \quad (14)$$

Then, this solution is unique with the energy estimates

$$\|(p,v)(t)\|_2^2 + \int_0^t \|\nabla p(\tau)\|_1^2 + \|\nabla v(\tau)\|_2^2 d\tau \le C\|(p_0,v_0)\|_2^2, \quad (15)$$

$$\|s(t)\|_2 \le C\|(p_0,v_0,s_0)\|_2 \exp\left(C\|(p_0,v_0)\|_2\right), \quad (16)$$

and the decay properties

$$\|\nabla(p,v)(t)\|_1 \le C_0(1+t)^{-1}, \quad (17)$$

$$\|(p,v)(t)\|_{L^q} \le C_0(1+t)^{1/q-1}, \quad 2\le q\le 4, \quad (18)$$

$$\|(p,v)(t)\|_{L^q} \le C_0(1+t)^{-((2+q)/2q)}, \quad 4 < q\le\infty, \quad (19)$$

$$\|\partial_t(p,v,s)(t)\|_{L^2} \le C_0(1+t)^{-(1/2)}. \quad (20)$$

The rest of paper is used to prove Proposition 4.

3. Energy Estimate

For later use, we introduce some useful analytic results.

Lemma 5. *Let $f\in H^2(\mathbb{R}^2)$; then, we have the following Sobolev inequalities:*

$$\|f\|_{L^\infty} \le \|f\|_{H^{1+\varepsilon}} \le \|f\|_2, \quad \forall\varepsilon > 0,$$
$$\|f\|_{L^4} \le \|f\|_{L^2}^{1/2}\|\nabla f\|_{L^2}^{1/2} \le \|f\|_1, \quad (21)$$
$$\|f\|_{L^q} \le \|f\|_1, 2\le q\le 4.$$

Lemma 6 (lower-order energy estimate for (p,v)). *Under the assumption of Proposition 4, there exists a $\delta_1 > 0$ arbitrarily small and independent of ε, such that*

$$\frac{1}{2}\frac{d}{dt}\left(\|(p,v)(t)\|_{L^2}^2 + 2\delta_1\langle\nabla p, v\rangle(t)\right) + C\|\nabla v(t)\|_{L^2}^2 + C\delta_1\|\nabla p(t)\|_{L^2}^2 \le C\varepsilon\left\|\left(v,\nabla^2 v\right)(t)\right\|_{L^2}^2 + C\delta_1\left\|\nabla^2 v(t)\right\|_{L^2}^2, \quad \forall 0\le t\le T. \quad (22)$$

Proof. Multiplying $(2.1)_1$, $(2.1)_2$ by p, v, respectively, integrating them over $\mathbb{R}^2$ and then adding them together, we obtain

$$\frac{1}{2}\frac{d}{dt}\|(p,v)\|_{L^2}^2 + \alpha_3\|\nabla v\|_{L^2}^2 + \alpha_4\|\nabla\cdot v\|_{L^2}^2 = \langle p,f\rangle + \langle v,g\rangle. \quad (23)$$

$\langle p,f\rangle$ and $\langle v,g\rangle$ can be estimated as follows. For the first term, Lemma 5 together with (14) and the Hölder inequality implies

$$\begin{aligned}\langle p,f\rangle &= -\frac{(1+c_v)\alpha_1}{c_v}\langle p, p\nabla\cdot v\rangle - \alpha_1\langle p, v\cdot\nabla p\rangle + \frac{1}{c_v}\langle p,\Psi(\alpha_1 v)\rangle \\ &= \left(-\frac{(1+c_v)\alpha_1}{c_v} + \frac{\alpha_1}{2}\right)\left\langle p^2,\nabla\cdot v\right\rangle + \frac{\alpha_1^2}{c_v}\langle p,\Psi(v)\rangle \\ &\le C\left(\|p\|_{L^4}^2\|\nabla v\|_{L^2} + \|p\|_{L^\infty}\|\nabla v\|_{L^2}^2\right) \\ &\le C\left(\|p\|_{L^2}\|\nabla p\|_{L^2}\|\nabla v\|_{L^2} + \|p\|_2\|\nabla v\|_{L^2}^2\right) \\ &\le C\varepsilon\left(\|\nabla p\|_{L^2}^2 + \|\nabla v\|_{L^2}^2\right). \end{aligned} \quad (24)$$

For the second term,

$$|\langle v, g\rangle| \leq C\left\{|\langle v,(v\cdot\nabla)v\rangle| + \left|\left\langle v, \left(\frac{1}{\rho}-\frac{1}{\rho_\infty}\right)\nabla p\right\rangle\right| + \left|\left\langle v, \left(\frac{1}{\rho}-\frac{1}{\rho_\infty}\right)\Delta v\right\rangle\right| + \left|\left\langle v, \left(\frac{1}{\rho}-\frac{1}{\rho_\infty}\right)\nabla\nabla\cdot v\right\rangle\right|\right\}. \tag{25}$$

Similar to the proof of (24), by Lemma 5, (14), the Hölder inequality, and the fact that

$$\frac{1}{\rho}-\frac{1}{\rho_\infty} \sim O(1)(p+s), \tag{26}$$

which is derived from (3) and (14), we have

$$\begin{aligned}
|\langle v,(v\cdot\nabla)v\rangle| &\leq C\|v\|_{L^4}^2\|\nabla v\|_{L^2} \leq C\varepsilon\|\nabla v\|_{L^2}^2,\\
\left|\left\langle v, \left(\frac{1}{\rho}-\frac{1}{\rho_\infty}\right)\nabla p\right\rangle\right| &\leq \|v\|_{L^2}\left\|\frac{1}{\rho}-\frac{1}{\rho_\infty}\right\|_{L^\infty}\|\nabla p\|_{L^2}\\
&\leq C\|v\|_{L^2}\|(p,s)\|_2\|\nabla p\|_{L^2}\\
&\leq C\varepsilon\left(\|v\|_{L^2}^2+\|\nabla p\|_{L^2}^2\right),\\
\left|\left\langle v, \left(\frac{1}{\rho}-\frac{1}{\rho_\infty}\right)\Delta v\right\rangle\right| &\leq \left\|\frac{1}{\rho}-\frac{1}{\rho_\infty}\right\|_{L^\infty}|\langle v,\Delta v\rangle|\\
&\leq C\varepsilon\|\nabla v\|_{L^2}^2,\\
\left|\left\langle v, \left(\frac{1}{\rho}-\frac{1}{\rho_\infty}\right)\nabla\nabla\cdot v\right\rangle\right| &\leq C\varepsilon\|\nabla v\|_{L^2}^2.
\end{aligned} \tag{27}$$

Therefore,

$$|\langle v, g\rangle| \leq C\varepsilon\left(\|(v,\nabla v)\|_{L^2}^2+\|\nabla p\|_{L^2}^2\right). \tag{28}$$

Hence, combining (23), (24), and (28) yields

$$\frac{1}{2}\frac{d}{dt}\|p,v\|_{L^2}^2 + C\|\nabla v\|_{L^2}^2 \leq C\varepsilon\left(\|v\|_{L^2}^2+\|\nabla p\|_{L^2}^2\right). \tag{29}$$

Next, we shall estimate $\|\nabla p\|_{L^2}^2$. Multiplying $(2.1)_2$ by ∇p and integrating them over $\mathbb{R}^2$, we get

$$\begin{aligned}
\alpha_2\|\nabla p\|_{L^2}^2 &= \langle -v_t,\nabla p\rangle + \alpha_3\langle\Delta v,\nabla p\rangle + \alpha_4\langle\nabla\nabla\cdot v,\nabla p\rangle + \langle g,\nabla p\rangle\\
&= -\frac{d}{dt}\langle v,\nabla p\rangle + \langle v,\nabla p_t\rangle + \alpha_3\langle\Delta v,\nabla p\rangle\\
&\quad+\alpha_4\langle\nabla\nabla\cdot v,\nabla p\rangle + \langle g,\nabla p\rangle\\
&= -\frac{d}{dt}\langle v,\nabla p\rangle - \langle\nabla\cdot v,p_t\rangle + \alpha_3\langle\Delta v,\nabla p\rangle\\
&\quad+\alpha_4\langle\nabla\nabla\cdot v,\nabla p\rangle + \langle g,\nabla p\rangle\\
&= -\frac{d}{dt}\langle v,\nabla p\rangle + \langle\alpha_2\nabla\cdot v - f,\nabla\cdot v\rangle + \alpha_3\langle\Delta v,\nabla p\rangle\\
&\quad+\alpha_4\langle\nabla\nabla\cdot v,\nabla p\rangle + \langle g,\nabla p\rangle,
\end{aligned} \tag{30}$$

where

$$\begin{aligned}
|\langle -f,\nabla\cdot v\rangle| &\leq C\varepsilon\|(v,\nabla v)\|_{L^2}^2,\\
|\langle g,\nabla p\rangle| &\leq C\varepsilon\left(\left\|\left(v,\nabla v,\nabla^2 v\right)\right\|_{L^2}^2+\|\nabla p\|_{L^2}^2\right).
\end{aligned} \tag{31}$$

which are based on the similar calculation and Young's inequality. In brief, we obtain

$$\frac{\alpha_2}{2}\|\nabla p\|_{L^2}^2 + \frac{d}{dt}\langle v,\nabla p\rangle \leq C\left\|\left(\nabla v,\nabla^2 v\right)\right\|_{L^2}^2 + C\varepsilon\|v\|_{L^2}^2. \tag{32}$$

Finally, multiplying (32) by δ_1 that is small but fixed and adding it to (29), we can derive (22). This completes the proof of the lemma. □

Next, we turn to estimate the higher-order energy for (p,v).

Lemma 7 (higher-order energy estimate for (p,v)). *Under the assumption of Lemma 6, there is a small enough but fixed $\delta_2>0$, which is independent of ε, such that*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\left\|\left(\nabla p,\nabla^2 p\right)(t)\right\|_{L^2}^2 + \left\|\left(\nabla v,\nabla^2 v\right)\right\|_{L^2}^2 + 2\delta_2\left\langle\nabla^2 p,\nabla v\right\rangle(t)\right)\\
&\quad+ C\left\|\left(\nabla^2 v,\nabla^3 v\right)(t)\right\|_{L^2}^2 + C\delta_2\left\|\nabla^2 p(t)\right\|_{L^2}^2\\
&\leq C\varepsilon(1+\delta_2)\|\nabla v(t)\|_{L^2}^2 + C\varepsilon\|\nabla p(t)\|_{L^2}^2, \quad \forall 0\leq t\leq T.
\end{aligned} \tag{33}$$

Proof. Applying ∇ to $(2.1)_1$, $(2.1)_2$ and multiplying by ∇p, ∇v, respectively, integrating them over $\mathbb{R}^2$ and using the same calculation technique as before, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\nabla p,\nabla v)\|_{L^2}^2 + \alpha_3\left\|\nabla^2 v\right\|_{L^2}^2\\
&\quad\leq \langle\nabla p,\nabla f\rangle + \langle\nabla v,\nabla g\rangle \leq \langle\nabla p,\nabla f\rangle + \langle\Delta v, g\rangle\\
&\quad\leq C\varepsilon\left(\left\|\left(\nabla v,\nabla^2 v\right)\right\|_{L^2}^2 + \|\nabla p\|_{L^2}^2\right).
\end{aligned} \tag{34}$$

Next, applying ∇^2 to $(2.1)_1$, $(2.1)_2$ and multiplying by $\nabla^2 p$, $\nabla^2 v$, respectively, we can deduce

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\left(\nabla^2 p,\nabla^2 v\right)\right\|_{L^2}^2 + \alpha_3\left\|\nabla^3 v\right\|_{L^2}^2\\
&\quad\leq \left\langle\nabla^2 p,\nabla^2 f\right\rangle + \left\langle\nabla^2 v,\nabla^2 g\right\rangle\\
&\quad\leq \left\langle\nabla^2 p,\nabla^2 f\right\rangle + \langle\Delta\Delta v, g\rangle,
\end{aligned} \tag{35}$$

where

$$|\langle\nabla^2 p, \nabla^2 f\rangle| \leq C\varepsilon\left(\left\|\left(\nabla^2 v, \nabla^3 v\right)\right\|_{L^2}^2 + \left\|\nabla^2 p\right\|_{L^2}^2\right),$$

$$\begin{aligned}|\langle\Delta\Delta v, g\rangle| \leq C\Bigg\{&|\langle\Delta\Delta v, (v\cdot\nabla)v\rangle| + \left|\left\langle\Delta\Delta v, \left(\frac{1}{\rho} - \frac{1}{\rho_\infty}\right)\nabla p\right\rangle\right| \\ &+ \left|\left\langle\Delta\Delta v, \left(\frac{1}{\rho} - \frac{1}{\rho_\infty}\right)\Delta v\right\rangle\right| \\ &+ \left|\left\langle\Delta\Delta v, \left(\frac{1}{\rho} - \frac{1}{\rho_\infty}\right)\nabla\nabla\cdot v\right\rangle\right|\Bigg\}.\end{aligned} \tag{36}$$

The terms on the right-hand side are estimated as below:

$$\begin{aligned}|\langle\Delta\Delta v, (v\cdot\nabla)v\rangle| &\leq C\|\nabla v\|_{L^4}^2\left\|\nabla^3 v\right\|_{L^2} \\ &\quad + C\left\|\nabla^3 v\right\|_{L^2}\left\|\nabla^2 v\right\|_{L^2}\|v\|_{L^\infty} \\ &\leq C\varepsilon\left\|\left(\nabla^2 v, \nabla^3 v\right)\right\|_{L^2}^2,\end{aligned}$$

$$\begin{aligned}\left|\left\langle\Delta\Delta v, \left(\frac{1}{\rho} - \frac{1}{\rho_\infty}\right)\nabla p\right\rangle\right| &\leq \left\|\left(\frac{1}{\rho} - \frac{1}{\rho_\infty}\right)\right\|_{L^\infty}\left|\left\langle\nabla\Delta v, \nabla^2 p\right\rangle\right| \\ &\leq C\varepsilon\left(\left\|\nabla^3 v\right\|_{L^2}^2 + \left\|\nabla^2 p\right\|_{L^2}^2\right),\end{aligned}$$

$$\left|\left\langle\Delta\Delta v, \left(\frac{1}{\rho} - \frac{1}{\rho_\infty}\right)\Delta v\right\rangle\right| \leq C\varepsilon\left\|\nabla^3 v\right\|_{L^2}^2,$$

$$\left|\left\langle\Delta\Delta v, \left(\frac{1}{\rho} - \frac{1}{\rho_\infty}\right)\nabla\nabla\cdot v\right\rangle\right| \leq C\varepsilon\left\|\nabla^3 v\right\|_{L^2}^2. \tag{37}$$

Thus,

$$|\langle\Delta\Delta v, g\rangle| \leq C\varepsilon\left(\left\|\left(\nabla^2 v, \nabla^3 v\right)\right\|_{L^2}^2 + \left\|\nabla^2 p\right\|_{L^2}^2\right), \tag{38}$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\left(\nabla^2 p, \nabla^2 v\right)\right\|_{L^2}^2 + C\left\|\nabla^3 v\right\|_{L^2}^2 \\ &\quad \leq C\varepsilon\left(\left\|\nabla^2 v\right\|_{L^2}^2 + \left\|\nabla^2 p\right\|_{L^2}^2\right).\end{aligned} \tag{39}$$

Now applying ∇ to $(2.1)_2$ and multiplying by $\nabla^2 p$, we have

$$\begin{aligned}\alpha_2\left\|\nabla^2 p\right\|_{L^2}^2 = &\left\langle-\nabla v_t, \nabla^2 p\right\rangle + \alpha_3\left\langle\nabla\Delta v, \nabla^2 p\right\rangle \\ &+ \alpha_4\left\langle\nabla^2\nabla\cdot v, \nabla^2 p\right\rangle + \left\langle\nabla g, \nabla^2 p\right\rangle.\end{aligned} \tag{40}$$

Similar to the estimate of $\|\nabla p\|_{L^2}^2$, we transform the formula $\langle-\nabla v_t, \nabla^2 p\rangle$ as below:

$$\begin{aligned}\left\langle-\nabla v_t, \nabla^2 p\right\rangle &= -\frac{d}{dt}\left\langle\nabla v, \nabla^2 p\right\rangle + \langle\nabla\cdot\Delta v, p_t\rangle \\ &= -\frac{d}{dt}\left\langle\nabla v, \nabla^2 p\right\rangle + \langle-\alpha_2\nabla\cdot v + f, \nabla\cdot\Delta v\rangle,\end{aligned} \tag{41}$$

and derive

$$\frac{\alpha_2}{2}\left\|\nabla^2 p\right\|_{L^2}^2 + \frac{d}{dt}\left\langle\nabla v, \nabla^2 p\right\rangle \leq C\left\|\left(\nabla^2 v, \nabla^3 v\right)\right\|_{L^2}^2 + C\varepsilon\|\nabla v\|_{L^2}^2. \tag{42}$$

Multiplying (42) by δ_2 that is small but fixed, and combining it with (34) and (39), we finally obtain (33). This completes the proof of the lemma. □

The lemma below gives the energy estimate for the entropy s.

Lemma 8. *For $0 \leq t \leq T$, it holds that*

$$\frac{d}{dt}\|s(t)\|_2^2 \leq C\|\nabla v(t)\|_2\|s(t)\|_2^2 + C\varepsilon\|\nabla v(t)\|_2^2. \tag{43}$$

Proof. For each multi-index α with $0 \leq |\alpha| \leq 2$, we apply ∂_x^α to $(2.1)_3$, multiply it by $\partial_x^\alpha s$, integrate it on $\mathbb{R}^2$, and then sum them up to deduce

$$\frac{1}{2}\frac{d}{dt}\|s\|_2^2 = -\alpha_1\sum_{0\leq|\alpha|\leq 2}\langle\partial_x^\alpha(v\cdot\nabla s), \partial_x^\alpha s\rangle + \sum_{0\leq|\alpha|\leq 2}\langle\partial_x^\alpha h, \partial_x^\alpha s\rangle = W_1 + W_2. \tag{44}$$

For W_1, one has

$$W_1 = -\alpha_1\sum_{0\leq|\alpha|\leq 2}\langle v\cdot\nabla\partial_x^\alpha s, \partial_x^\alpha s\rangle - \alpha_1\sum_{\substack{0\leq|\beta|\leq|\alpha|-1 \\ 1\leq|\alpha|\leq 2}}\left\langle\partial_x^{\alpha-\beta}v\cdot\nabla\partial_x^\beta s, \partial_x^\alpha s\right\rangle, \tag{45}$$

where, for each α with $0 \leq |\alpha| \leq 2$, one can infer that

$$\langle v\cdot\nabla\partial_x^\alpha s, \partial_x^\alpha s\rangle \leq \|\nabla\cdot v\|_{L^\infty}\|\partial_x^\alpha s\|_{L^2}^2 \leq \|\nabla v\|_2\|\partial_x^\alpha s\|_{L^2}^2, \tag{46}$$

and for each α, β with $0 \leq |\beta| \leq |\alpha| - 1$, $1 \leq |\alpha| \leq 2$, and $\beta \leq \alpha$, it holds that

$$\left\langle\partial_x^{\alpha-\beta}v\cdot\nabla\partial_x^\beta s, \partial_x^\alpha s\right\rangle \leq C\|\nabla v\|_2\|\nabla s\|_1^2. \tag{47}$$

Therefore, we have

$$W_1 \leq C\|\nabla v\|_2\|s\|_2^2. \tag{48}$$

Now, we turn to estimate W_2; using (14), we obtain

$$W_2 \le \sum_{0\le|\alpha|\le2} \|\partial_x^\alpha h\|_{L^2} \|\partial_x^\alpha s\|_{L^2} \le \varepsilon \sum_{0\le|\alpha|\le2} \left\| \partial_x^\alpha \left(\frac{\Psi(\alpha_1 v)}{p+p_\infty} \right) \right\| \le \varepsilon \sum_{\substack{\beta\le\alpha \\ 0\le|\alpha|\le2}} C_\alpha^\beta \left\| \partial_x^{\alpha-\beta} \left(\frac{1}{p+p_\infty} \right) \partial_x^\beta \Psi(\alpha_1 v) \right\| \le C\varepsilon \|\nabla v\|_2^2. \tag{49}$$

Combining (44) with (48) and (49) yields (43). □

Now, we are in the position to prove the energy estimate in Proposition 4.

Proof of (15) and (16). The first step is to prove (15), energy estimate for (p, v). By defining

$$\begin{aligned} E(t) &:= \|(p,v)\|_2^2 + 2\delta_1 \langle \nabla p, v\rangle + 2\delta_2 \langle \nabla^2 p, \nabla v\rangle \sim \|(p,v)\|_2^2, \\ F(t) &:= \|(p,v)\|_2^2 + \int_0^t \|\nabla p\|_1^2 + \|\nabla v\|_2^2 d\tau, \end{aligned} \tag{50}$$

and adding (22) and (33) together, one has

$$\frac{1}{2}\frac{d}{dt}E(t) + C\|\nabla v\|_2^2 + C\delta_1 \|\nabla p\|_1^2 \le C\varepsilon(1+\delta_1)\|v\|_{L^2}^2. \tag{51}$$

Integrating the above equation directly in time gives

$$\begin{aligned} F(t) &\le C\|(p_0, v_0)\|_2^2 + C\varepsilon(1+\delta_1)\int_0^t \|v\|_{L^2}^2 d\tau \\ &\le C\|(p_0, v_0)\|_2^2 + C\varepsilon T(1+\delta_1) \sup_{\tau\in[0,t]} \|v\|_{L^2}^2. \end{aligned} \tag{52}$$

Hence,

$$\sup_{\tau\in[0,t]} \|v\|_{L^2}^2 + \sup_{\tau\in[0,t]} \left(F(t) - \|v\|_{L^2}^2\right) \le C\|(p_0, v_0)\|_2^2 + C\varepsilon T(1+\delta_1) \sup_{\tau\in[0,t]} \|v\|_{L^2}^2, \tag{53}$$

which implies

$$F(t) \le C\|(p_0, v_0)\|_2^2, \tag{54}$$

when ε is small enough. In other words, (15) is valid.

The second step is to prove (16), the energy estimate for s. Summing up (22), (33), and (43) gives

$$\frac{d}{dt}y(t) \le C\|\nabla v\|_2 \|s\|_2^2 + C\varepsilon \|v\|_{L^2}^2, \tag{55}$$

where

$$y(t) = \|(p,v,s)\|_2^2 + \delta_1 \left(\langle v, \nabla p\rangle + \langle \nabla v, \nabla^2 p\rangle \right) \sim \|(p,v,s)\|_2^2. \tag{56}$$

Therefore,

$$\frac{d}{dt}y(t) \le (C\|\nabla v\|_2 + C\varepsilon) y(t). \tag{57}$$

By Grönwall's inequality and (15), we obtain

$$\begin{aligned} y(t) &\le y(0) \exp\left\{ \int_0^t (C\|\nabla v\|_2 + C\varepsilon) d\tau \right\} \\ &\le y(0) \exp\left\{ \int_0^t \left(\frac{C\|\nabla v\|_2^2}{2\varepsilon} + \frac{3C\varepsilon}{2} \right) d\tau \right\} \\ &\le y(0) \exp\left\{ CT\varepsilon + \frac{C\|(p_0, v_0)\|_2^2}{\varepsilon} \right\} \\ &\le Cy(0) \exp\left\{ C\|(p_0, v_0)\|_2 \right\}, \end{aligned} \tag{58}$$

which implies (16).

4. Decay Rates

In this section, we prove the decay rates in Proposition 4.

Proof of (17)–(20). Letting

$$\begin{aligned} G(t) := &\left\| \left(\nabla p, \nabla^2 p\right)(t) \right\|_{L^2}^2 + \left\| \left(\nabla v, \nabla^2 v\right)(t) \right\|_{L^2}^2 \\ &+ 2\delta_2 \left\langle \nabla^2 p, \nabla v \right\rangle (t) \sim \|\nabla (p,v)\|_1^2, \end{aligned} \tag{59}$$

we rewrite (33) as follows:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}G(t) + C\left\| \left(\nabla^2 v, \nabla^3 v\right)(t) \right\|_{L^2}^2 + C\delta_2 \left\| \nabla^2 p(t) \right\|_{L^2}^2 \\ &\quad \le C\varepsilon(1+\delta_2)\|\nabla v(t)\|_{L^2}^2 + C\varepsilon \|\nabla p(t)\|_{L^2}^2, \quad \forall 0 \le t \le T. \end{aligned} \tag{60}$$

Next, by adding $\|\nabla(p,v)\|_{L^2}^2$ to both sides of the above inequality, we deduce that for some constant $\alpha > 0$,

$$\frac{d}{dt}G(t) + \alpha G(t) \le C\|\nabla (p,v)\|_{L^2}^2. \tag{61}$$

Therefore,

$$G(t) \le G(0)e^{-\alpha t} + C\int_0^t e^{-\alpha(t-\tau)} \|\nabla (p,v)\|_{L^2}^2 d\tau. \tag{62}$$

To deal with $\|\nabla(p,v)\|_{L^2}^2$, we rewrite the solution of system (9) as

$$H(t) = e^{-t\mathbb{A}} H(0) + \int_0^t e^{-(t-\tau)\mathbb{A}} (f,g)(\tau) d\tau, \tag{63}$$

where $H(t) = (p(t), v(t))$ and $\mathbb{A}$ is a matrix-valued differential operator given by

$$\mathbb{A} = \begin{pmatrix} 0 & \alpha_2 \operatorname{div} \\ \alpha_2 \nabla & -\alpha_3 \Delta - \alpha_4 \nabla \operatorname{div} \end{pmatrix}, \tag{64}$$

and the solution semigroup $e^{-t\mathbb{A}}$ has the following property (see [31, 32]). □

Lemma 9. *Let $k > 0$ be an integer; then, $\forall t \geq 0$,*

$$\left\| \nabla^k e^{-t\mathbb{A}} H(0) \right\|_{L^2} \leq C(1+t)^{-((1+k)/2)} \left(\|H(0)\|_{L^1} + \|H(0)\|_1 \right). \tag{65}$$

Then, gathering (63) and (65), one has

$$\|\nabla(p,v)\|_{L^2} \leq CK_0(1+t)^{-1} + C\int_0^t (1+t-\tau)^{-1} \|(f,g)(\tau)\|_{L^1\cap H^1} d\tau, \tag{66}$$

where $K_0 = \|(p_0, v_0)\|_{H^2\cap L^1}$ and the nonlinear source can be estimated as follows:

$$\begin{aligned} &\|(f,g)(t)\|_{L^1} \leq C\varepsilon \left(\|\nabla p(t)\|_{L^2} + \|\nabla v(t)\|_1 \right), \\ &\|(f,g)(t)\|_1 \leq C\varepsilon \left(\|\nabla(p,v)(t)\|_1 + \left\|\nabla^3 v(t)\right\|_{L^2} \right). \end{aligned} \tag{67}$$

Hence,

$$\|\nabla(p,v)\|_{L^2} \leq CK_0(1+t)^{-1} + C\int_0^t (1+t-\tau)^{-1} \left(\|\nabla(p,v)\|_1 + \left\|\nabla^3 v\right\|_{L^2} \right) d\tau. \tag{68}$$

Now, if we define $M(t) = \sup_{0\leq\tau\leq t} (1+\tau)^2 G(\tau)$, then

$$\|\nabla(p,v)(\tau)\|_1 \leq C\sqrt{G(\tau)} \leq C(1+\tau)^{-1}\sqrt{M(t)}, \quad 0 \leq \tau \leq t, \tag{69}$$

which implies that

$$\|\nabla(p,v)(t)\|_1 \leq C(1+t)^{-1}\sqrt{M(t)}, \quad 0 \leq t \leq T. \tag{70}$$

Combining (68) and (70) and letting ε be small enough, we can deduce that

$$\begin{aligned} \|\nabla(p,v)\|_{L^2} &\leq CK_0(1+t)^{-1} + C\varepsilon\int_0^t (1+t-\tau)^{-1}(1+\tau)^{-1}\sqrt{M(\tau)} d\tau \\ &\quad + C\varepsilon\int_0^t (1+t-\tau)^{-1}\left\|\nabla^3 v\right\|_{L^2} d\tau \leq CK_0(1+t)^{-1} \\ &\quad + C\sqrt{\varepsilon}\sqrt{M(t)}\left(\sqrt{\varepsilon}\frac{\ln(1+T)}{1+t}\right) \\ &\quad + C\varepsilon\left(\int_0^t (1+t-\tau)^{-2} d\tau\right)^{1/2}\left(\int_0^t \left\|\nabla^3 v\right\|_{L^2}^2 d\tau\right)^{1/2} \\ &\leq C(1+t)^{-1}\left(K_0 + \sqrt{\varepsilon}\sqrt{M(t)}\right) + C\varepsilon(1+t)^{-1}. \end{aligned} \tag{71}$$

Substituting (71) into (62), one has

$$\begin{aligned} G(t) &\leq G(0)e^{-\alpha t} + C\left(K_0^2 + \varepsilon M(t) + \varepsilon^2\right)\int_0^t e^{-\alpha(t-\tau)}(1+\tau)^{-2} d\tau \\ &\leq C(1+t)^{-2}\left(G(0) + K_0^2 + \varepsilon M(t) + \varepsilon^2\right). \end{aligned} \tag{72}$$

Now by the definition of $M(t)$, we obtain

$$M(t) \leq C\left(G(0) + K_0^2 + \varepsilon M(t) + \varepsilon^2\right), \quad \text{for any } 0 \leq t \leq T. \tag{73}$$

Letting $\varepsilon > 0$ be small enough, the inequality above gives us

$$M(t) \leq C\left(G(0) + K_0^2\right) \leq CK_0^2. \tag{74}$$

Thus, (70) gives

$$\|\nabla(p,v)\|_1 \leq C_0(1+t)^{-1}, \quad 0 \leq t \leq T. \tag{75}$$

Meanwhile, applying (65) with $k = 0$ to (14), by (15) and (17), one can infer that

$$\begin{aligned} \|(p,v)\|_{L^2} &\leq CK_0(1+t)^{-(1/2)} + C\int_0^t (1+t-\tau)^{-(1/2)}\|(f,g)\|_{L^1\cap H^1} d\tau \\ &\leq CK_0(1+t)^{-(1/2)} + C\varepsilon K_0\int_0^t (1+t-\tau)^{-1}(1+\tau)^{-1} d\tau \\ &\quad + C\varepsilon\int_0^t (1+t-\tau)^{-(1/2)}\left\|\nabla^3 v\right\|_{L^2} d\tau \\ &\leq C_0(1+t)^{-(1/2)}, \quad 0 \leq t \leq T. \end{aligned} \tag{76}$$

Then, by Sobolev's inequality, (75), and (76), we get

$$\begin{aligned} &\|(p,v)\|_{L^\infty} \leq \|(p,v)\|_2 \leq C_0(1+t)^{-(1/2)}, \quad 0 \leq t \leq T, \\ &\|(p,v)\|_{L^4} \leq \|(p,v)\|_{L^2}^{1/2}\|\nabla(p,v)\|_{L^2}^{1/2} \leq C_0(1+t)^{-(3/4)}, \quad 0 \leq t \leq T. \end{aligned} \tag{77}$$

Thus, by interpolation inequality, one has

$$\|(p,v)\|_{L^q} \leq \|(p,v)\|_{L^2}^{(4-q)/q}\|(p,v)\|_{L^4}^{(2q-4)/q} \leq C_0(1+t)^{1/q-1}, \tag{78}$$

for any $2 \leq q \leq 4$, $0 \leq t \leq T$, and

$$\|(p,v)\|_{L^q} \leq \|(p,v)\|_{L^\infty}^{1-4/q}\|(p,v)\|_{L^4}^{4/q} \leq C_0(1+t)^{-((2+q)/2q)}, \tag{79}$$

for any $4 < q \leq \infty$, $0 \leq t \leq T$. Therefore, the optimal decay rates (17)–(19) have been proven.

We use the estimates above and (9) to deduce that

$$\begin{aligned}\|\partial_t(p,v,s)\|_{L^2} &\le \|\alpha_2\nabla\cdot v\|_{L^2} + \|f\|_{L^2} + \|\alpha_2\nabla p - \alpha_3\Delta v - \alpha_4\nabla\nabla\cdot v\|_{L^2} \\ &\quad + \|g\|_{L^2} + \|\alpha_1(v\cdot\nabla s)\|_{L^2} + \|h\|_{L^2} \\ &\le C_0(1+t)^{-(1/2)}, \quad 0\le t\le T. \end{aligned} \tag{80}$$

Therefore, (20) is finally proven.

5. Conclusion

The motivation of this paper is to refine the previous works of [3, 5]. As mentioned above, to prove the global existence of the compressible Navier-Stokes equations, ref. [3] needs to use some complicated techniques based on the notions of the homogeneous Besov space and the hybrid Besov space to remove a condition in [5]. However, in this paper, we use a much simpler method to achieve this, to complete a prior estimate on entropy $\mathcal{S}$ which is important to prove the global existence of the compressible Navier-Stokes equations by some simple analysis.

The results are in the H^2-framework; it is possible to consider similar problems in functional space with lower regularity.

Data Availability

All data generated or analysed during this study are included in this published article.

Disclosure

The manuscript has been submitted as a preprint according to the following link: "https://www.authorea.com/users/488979/articles/572918-global-existence-and-decay-estimate-for-the-2-d-compressible-navier-stokes-equations-without-heat-conductivity."

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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