Open Access Library Journal
2023, Volume 10, e10966
ISSN Online: 2333-9721
ISSN Print: 2333-9705

Dynamical Localization of the Quasi-Periodic Schrödinger Operators

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How to cite this paper: Refai, W. (2023) Dynamical Localization of the Quasi-Periodic Schrödinger Operators. *Open Access Library Journal*, **10**: e10966.
https://doi.org/10.4236/oalib.1110966

Received: November 6, 2023
Accepted: December 26, 2023
Published: December 29, 2023

Abstract

In this paper, we study the spectral properties of a family of discrete one-dimensional quasi-periodic Schrödinger operators (depending on a phase theta). In the perturbative regime and in large disorder, under some conditions on v and a diophantine rotation number, we prove by using KAM theory that this operator satisfies both Anderson and dynamical localization for all $\theta \in [0, 2\pi)$.

Subject Areas

Dynamical System, Functional Analysis

Keywords

Quasi-Periodic Schrödinger Operators, Pure Point Spectrum, Eigenfunctions, Dynamical Localization

1. Introduction

The spectral theory of Schrödinger operators with random or almost periodic potentials has been an area of very active study since the late 1970's. From the beginning, it has been understood and emphasized that these two classes of models share an important property, namely that the potentials can be generated dynamically. On the one hand, this makes a unified proof of basic spectral results possible, such as the almost sure constancy of the spectrum and the spectral type, since they hold as soon as the dynamical framework is fixed and an ergodic measure is chosen. On the other hand, by the very nature of the dynamical definition of the potentials, it comes as no surprise that tools from dynamics will enter the spectral analysis of these operators.

1.1. Quasi-Periodic Schrödinger Operator on $\ell^2(\mathbb{Z})$

Consider the one-dimensional quasi-periodic Schrödinger operator

$$(H_\theta u)_n =: -\varepsilon(u_{n+1}+u_{n-1})+v(\theta+n\omega)u_n,\ n\in\mathbb{Z}, \tag{1}$$

where ω is a real number and v is a smooth function on $[0,2\pi)$.

We may assume the following on the data:

- Diophantine condition on the frequency ω: That is:

$$\|n\omega\| := \inf_{m\in\mathbb{Z}}|n\omega-2\pi m| \geq \frac{\kappa}{|n|^\tau}\quad \forall n\in\mathbb{Z}\setminus\{0\}, \tag{2}$$

for some constants $\kappa>0$ and $\tau>1$.

- v is a function of class $\mathcal{C}^1$, satisfies:

$$0<\alpha\leq|\partial_\theta v(\theta)|\leq c<\infty,\ \forall\theta. \tag{3}$$

1.2. Anderson Localization

We say that an operator satisfies Anderson localization if it has pure point spectrum with exponentially decaying eigenfunctions.

1.3. Dynamical Localization

Another localization criterion stronger than Anderson localization, this is called dynamical localization. Consider the evolution equation in time associated to H_θ,

$$i\frac{\partial\psi}{\partial t}=H_\theta\psi,\ \psi_\theta(t)=\mathrm{e}^{-itH_\theta}\psi \tag{4}$$

where $\psi\in\ell^2(\mathbb{Z})$. We say that H_θ satisfies,

- The dynamical localization (D. L), if for a.e θ,

$$\sup_t\left(\sum_{n\in\mathbb{Z}}(1+n^2)|\psi_\theta(t,n)|^2\right)<+\infty. \tag{5}$$

- The strong dynamical localization (Strong D.L), if,

$$\sup_t\int_{[0,2\pi)}\left(\sum_{n\in\mathbb{Z}}(1+n^2)|\psi_\theta(t,n)|^2\right)\mathrm{d}\theta<+\infty. \tag{6}$$

The main result of this paper is the following:

Under the assumptions ((2) and (3)), we prove the following:

Theorem 1. 1) Assume that ω and v are as above, then there exists a constant $\varepsilon_0=\varepsilon_0(\alpha,\kappa,\tau)$ such that:

If $|\varepsilon|<\varepsilon_0$ then H_θ is pure point with a set of exponential decaying eigenfunctions which form an orthonormal basis of $\ell^2(\mathbb{Z})$ for all θ.

2) Assume that (2) and (3) are hold, then for a.e ω the operator H_θ satisfies the strong dynamical localization (D.L) for all θ.

Remark.

This result improves in some way the previous one by Eliasson, such that dynamical localization is proven with an appropriate potential. To my knowledge, there are no results on the spectral properties of Schrödinger operators with discontinuous potential. The ideas presented in this paper can be used to obtain new results for several models with discontinuities.

Let us review now, some of the results in the literature that are most relevant to this paper:

- In [1], L.H. Eliasson considered the operator H_θ given by (1) with frequency ω satisfying a Diophantine condition and the function v satisfying a Gevrey-class regularity and a transversality condition. Under these assumptions, he proved using KAM methods that for $|\varepsilon|<\varepsilon_0$ where ε_0 depends on the function v and on the Diophantine condition on ω the operator H_θ has pure point spectrum for a.e. $\theta \in \mathbb{T}$. Moreover, this implies, using Kotani's theory (see [2]) that the Lyapunov exponent is nonzero for a.e. energy *E*. The author has also suggested that the argument could be modified to obtain exponential decay of the eigenfunctions, but without proof.
- J. Bourgain and M. Goldstein considered (see [3]) the operator H_θ given by (1) where ω satisfies a Diophantine condition and v is a non-constant analytic function. They also assumed that the Lyapunov exponent is positive for a.e. ω and for all *E*. The authors proved that the operator H_θ satisfies Anderson localization with exponential decay of the eigenfunctions at almost Lyapunov rate for every θ and for a.e. ω. Their result is nonperturbative -the constant ε_0 depends only on the potential *v*. In this paper we use the KAM approach which is a perturbative method—the constant ε_0 depends on v and ω -with different conditions on *v*, also we prove the Dynamical localization which is stronger than Anderson localization.
- For the quasi-periodic model, and unlike Anderson's case, there were fewer results that were found for this kind of localization. However, several results on the (D.L.) were published for the random model, for more references see [4] [5].

In the case of quasi-periodic models, this localization phenomenon (D.L) implies Anderson localization, and which also implies by the RAGE theorem that the spectrum is purely punctual (see [6]). In view of this, these models are natural candidates for (D.L). In this context, F. Germinet and S. Jitomirskaya (see [7]), have improved the results of [8] and [9], by proving the strong (D.L) of the operator $-\Delta+\lambda\cos\left(2\pi\left(\theta+n\omega\right)\right)$, for all $\lambda>2$ and diophantine ω. Later, in 2004, J. Bourgain and S. Jitomirskaya announced (without demonstration) this result for the quasi-periodic Schrödinger operators, see [10] for more details.

- Quasi-periodic operators have been heavily studied over the years; we direct the reader to the survey [11] for a guide on the literature.

1.4. Idea of Proof

1.4.1. KAM Theory

KAM theory is the perturbative theory, initiated by Kolmogorov, Arnold and Moser in the 1950s, of quasi-periodic motions in conservative dynamical systems. This theory deals with the persistence, under perturbation, of quasi-periodic motions in Hamiltonian dynamical systems. An important example is given by the dynamics of nearly integrable Hamiltonian systems. In general, the phase space of a completely integrable Hamiltonian system of n degrees of freedom is foliated by invariant n-dimensional tori (possibly of different topology). KAM theory shows that, under suitable regularity and non-degeneracy assumptions, most (in measure theoretic sense) of such tori persist (slightly deformed) under small Hamiltonian perturbations. The union of persistent n-dimensional tori (Kolmogorov set) tends to fill the whole phase space as the strength of the perturbation is decreased. The major technical problem arising in this context is due to the appearance of resonances and small divisors in the associated formal perturbation series.

1.4.2. Application to the Schrödinger Operators

The method of proof is a refinement of an already refined KAM method developed by Eliasson in a series of fundamental papers in the theory of quasi-periodic Schrodinger operators (especially [1]). The method consists of an infinite sequence of transformations aiming at conjugating the infinite dimensional matrix defined by the operator on $\ell^2(\mathbb{Z})$:

$$D(\theta)+\varepsilon F(\theta)=\begin{pmatrix} \ddots & & & & 0 \\ & v(\theta-\omega) & -\varepsilon & & \\ & -\varepsilon & v(\theta) & -\varepsilon & \\ & & -\varepsilon & v(\theta+\omega) & \\ 0 & & & & \ddots \end{pmatrix}$$

to a diagonal matrix $D_\infty(\theta,\varepsilon)$, by an orthogonal matrix made up of a complete set of eigenvectors. An iterative procedure that permits us to construct a such matrix,

$$U_j^*\cdots U_1^*(D+\varepsilon F)U_1\cdots U_j=D_{j+1}+F_{j+1}$$

that conjugate $D+\varepsilon F$ closer and closer to a diagonal matrix $D_j=\operatorname{diag}\left(v_j(\theta+k\omega)\right)$.

In the perturbative regime, these matrices are perturbations of diagonal matrices and the problem is to diagonalize them completely or partially, *i.e.* to show that they have some point spectrum. The unperturbed matrices have a dense point spectrum so that their eigenvalues are, up to any order of approximation, of infinite multiplicity, which is a very delicate situation to perturb. For matrices with strong decay of the off-diagonal elements, this difficulty can be overcome if the eigenvectors are sufficiently well clustering. One way to handle this is to control the almost multiplicities of the eigenvalues. The eigenvalues are given by functions of one or several parameters and in order to control the almost multiplicities it is necessary that these functions are not too flat. If the parameter space is one-dimensional and if the quasi-periodic frequencies satisfy some Diophantine condition, then it turns out that this control of the derivatives of eigenvalues is not only necessary, but also sufficient for the control of the almost multiplicities. If the parameter space is higher-dimensional this control is more difficult to achieve and not yet well understood.

2. Iterative Study

This section is organized in the following way:

- A first part devoted to the study of the first step of the iteration described in the previous paragraph. Under some conditions on v and ω we construct the matrices U_1, F_2 and D_1 which satisfy the estimates of Lemma 2.
- In the second part and after a suitable choice of parameters, an inductive proposition, Proposition 3 is introduced in order to prove the first result in Theorem 1, which is a simple consequence of Lemma 4.
- At the end we give the proof of Theorem 1(2).

Consider now the symmetric infinite-dimensional matrix that depends on the parameter θ, $D(\theta)+F(\theta)$ with,

$$D(\theta)=\begin{pmatrix} \ddots & & & & 0 \\ & v(\theta-\omega) & & & \\ & & v(\theta) & & \\ & & & v(\theta+\omega) & \\ 0 & & & & \ddots \end{pmatrix}.$$

For the formulation of the first step of iteration we shall assume the following:

- The rotation number ω and the potential v satisfy (2) and (3).
- $\begin{cases} \left|F_i^j\right| \le \varepsilon \mathrm{e}^{-|i-j|\rho} \text{ with } \rho>0 \\ \left|\partial_\theta F_i^j\right| \le \sqrt{\varepsilon}. \end{cases}$.
- Consider the equation:

$$\mathrm{e}^{-X}(D+F)\mathrm{e}^{X}=D'+F' \tag{7}$$

where the matrices X, D' and F' are defined in the following way:

Let $N:=\dfrac{1}{\varepsilon^a \rho}$ for $0<a<\dfrac{1}{4\tau}$.

1) The matrix X is defined by $\begin{cases} X_i^j=0 \text{ if } i=j \text{ or } |i-j|>N \\ X_i^j=-\dfrac{F_i^j}{v_i-v_j} \text{ otherwise} \end{cases}$ and satisfies the equation:

$$[D,X]=F^N-D'+D \tag{8}$$

where $\left(F^N\right)_i^j=\begin{cases} F_i^j \text{ if } |i-j|\le N \\ 0 \text{ otherwise} \end{cases}$.

2) $(D'-D)_i^i=F_i^i$.

3) $F'(\theta)=\mathrm{e}^{-X(\theta)}\left(D(\theta)+F(\theta)\right)\mathrm{e}^{X(\theta)}-D'(\theta)$.

Lemma 2. Let $0<a\tau+b<\dfrac{1}{4}$ and $\sigma=\varepsilon^b$.

If $\varepsilon \le \left(\frac{\kappa}{2^6}\alpha\rho^{\tau+1}\right)^{\frac{1}{1-(a\tau+b)}}$ then:

1)

a) $\left|X_i^j\right| \le \varepsilon \frac{N^\tau}{\alpha\kappa} \mathrm{e}^{-|i-j|\rho} := A\mathrm{e}^{-|i-j|\rho}$.

b) $\|X\| \le \varepsilon \frac{N^\tau}{\alpha\kappa} \frac{2}{1-\mathrm{e}^{-\rho}}$.

c) $\left|\left(\mathrm{e}^{\pm X} - I\right)_i^j\right| \le \frac{2^7 A}{\sigma\rho} \mathrm{e}^{-|i-j|\rho'}$ where $\rho' = \rho - \frac{\sigma}{2}\rho$.

2)

a) $\left|F_i'^j\right| \le 16\left(\frac{2^5 A}{\sigma\rho}\right)^2 \mathrm{e}^{-|i-j|\rho'}$.

b) $\|F'\| \le 16\left(\frac{2^5 A}{\sigma\rho}\right)^2 \frac{2}{1-\mathrm{e}^{-\rho'}} := \varepsilon' \frac{2}{1-\mathrm{e}^{-\rho'}}$.

3)

a) $\left|\partial_\theta v'(\theta)\right| \ge \alpha - \sqrt{\varepsilon} := \alpha'$ where $D'(\theta) = \mathrm{diag}\left(v'(\theta + n\omega)\right)$.

b) $\left|\partial_\theta F_i'^j(\theta)\right| \le \sqrt{\varepsilon'}$.

c) $\left|\partial_\theta v'(\theta)\right| < c + \sqrt{\varepsilon} := c'$.

4) $\left|(D' - D)_i^i\right| \le \varepsilon$.

Proof.

1(a): Let $i \ne j$ and $|i-j| \le N$, we have:

$$X_i^j = -\frac{F_i^j}{v_i - v_j} \quad \text{therefore} \quad \left|X_i^j\right| \le \frac{\varepsilon}{\left|v_i - v_j\right|} \mathrm{e}^{-|i-j|\rho}$$

since $\left|v_i - v_j\right| = \left|v(\theta + i\omega) - v(\theta + j\omega)\right| \ge \inf\left|\partial_\theta v\right| \left|(i-j)\omega\right| \ge \frac{\alpha\kappa}{|i-j|^\tau}$, then it follows that

$$\left|X_i^j\right| \le \varepsilon \frac{N^\tau}{\alpha\kappa} \mathrm{e}^{-|i-j|\rho}.$$

1(b): Using 1(a) we obtain:

$$\sum_{i\in\mathbb{Z}} \left|X_i^j\right| \le A \sum_{i\in\mathbb{Z}} \mathrm{e}^{-|i-j|\rho} \le A \sum_{i\in\mathbb{Z}} \mathrm{e}^{-|i|\rho} \le \frac{2A}{1-\mathrm{e}^{-\rho}} \tag{9}$$

Thus from the generalized Young inequality [12] (page 9), (9) implies an estimate of X in the operator norm on $\ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$.

1(c): By lemma A_8 (Eliasson [13]) and for all $n \in \mathbb{N}$ we deduce that:

$$\left|(\underbrace{X \cdots X}_{n \text{ times}})_i^j\right| \le \left(\frac{2^5 A}{\sigma\rho}\right)^n \mathrm{e}^{-|i-j|\rho'},$$

hence $\left|\left(\mathrm{e}^{\pm X} - I\right)_i^j\right| \le \frac{2^7 A}{\sigma\rho} \mathrm{e}^{-|i-j|\rho'}$.

2(a): Let $\tilde{F}_N := F - F^N$ then we have:

$$
\begin{aligned}
F' &= \mathrm{e}^{-X}\left(D+F\right)\mathrm{e}^{X} - D' \\
&= \mathrm{e}^{-X}\left(D+F^{N}\right)\mathrm{e}^{X} + \mathrm{e}^{-X}\tilde{F}_{N}\mathrm{e}^{X} - D' \\
&= \left[X, F^{N}\right] - XDX - XF^{N}X + \sum_{\substack{m+n\geq 2\\ (m,n)\neq(1,1)}} \frac{(-X)^{n}}{n!}\left(D+F^{N}\right)\frac{X^{m}}{m!} \\
&\quad + \sum_{m+n\geq 0} \frac{(-X)^{n}}{n!}\tilde{F}_{N}\frac{X^{m}}{m!}.
\end{aligned}
$$

Now we have to estimate the elements of all matrices which constitute the matrix F'.

(*)1

$$
\begin{aligned}
\left|\left(\left[X, F^{N}\right]\right)_{i}^{j}\right| &\leq 2\sum_{k\in I}\left|X_{i}^{k}\right|\left|\left(F^{N}\right)_{k}^{j}\right| \\
&\leq 2\varepsilon A\sum_{k\in I}\mathrm{e}^{-(|k-i|+|k-j|)\rho} \\
&\leq 2\varepsilon A\sum_{k\in I}\mathrm{e}^{-(|k-i|+|k-j|)\rho'}\mathrm{e}^{-(|k-i|+|k-j|)\frac{\sigma\rho}{2}} \\
&\leq \varepsilon\frac{2^{5}A}{\sigma\rho}\mathrm{e}^{-|i-j|\rho'}
\end{aligned}
$$

where $I = \left\{k\in\mathbb{Z}; |k-i|\leq N \text{ and } |k-j|\leq N\right\}$.

In the same way we get:

(*)2

$$
\left|\left(XDX\right)_{i}^{j}\right| \leq \sum_{k\in I}\left|X_{i}^{k}\right|\left|\left(DX\right)_{k}^{j}\right| \leq \left(\frac{2^{5}A}{\sigma\rho}\right)^{2}\mathrm{e}^{-|i-j|\rho'}.
$$

(*)3

$$
\left|\left(XFX\right)_{i}^{j}\right| \leq \left(\frac{2^{5}A}{\sigma\rho}\right)^{2}\mathrm{e}^{-|i-j|\rho'}.
$$

(*)4

$$
\left|\left(\sum_{\substack{m+n\geq 2\\ (m,n)\neq(1,1)}} \frac{(-X)^{n}}{n!}\left(D+F^{N}\right)\frac{X^{m}}{m!}\right)_{i}^{j}\right| \leq 3\left(\frac{2^{5}A}{\sigma\rho}\right)^{2}(1+\varepsilon)\mathrm{e}^{-|i-j|\rho'}.
$$

(*)5

$$
\begin{aligned}
\left|\left(\sum_{m+n\geq 0}\frac{(-X)^{n}}{n!}\tilde{F}_{N}\frac{X^{m}}{m!}\right)_{i}^{j}\right| &\leq \sum_{m+n\geq 0}\frac{1}{n!m!}\sum_{k\in\mathbb{Z}}\left|\left(X^{n}\right)_{i}^{k}\right|\left|\left(\tilde{F}_{N}X^{m}\right)_{k}^{j}\right| \\
&\leq \sum_{m+n\geq 0}\frac{1}{n!m!}\sum_{k\in\mathbb{Z}}\left|\left(X^{n}\right)_{i}^{k}\right|\sum_{|k-\ell|>N}\left|\left(\tilde{F}_{N}\right)_{k}^{\ell}\right|\left|\left(X^{m}\right)_{\ell}^{j}\right| \\
&\leq \varepsilon\mathrm{e}^{-\frac{1}{\varepsilon^{a}}}\sum_{m+n\geq 0}\frac{1}{n!m!}\sum_{k\in\mathbb{Z}}\left|\left(X^{n}\right)_{i}^{k}\right|\sum_{|k-\ell|>N}\left|\left(X^{m}\right)_{\ell}^{j}\right| \\
&\leq \frac{3\varepsilon\mathrm{e}^{-\frac{1}{\varepsilon^{a}}}}{(\sigma\rho)^{2}}\mathrm{e}^{-|i-j|\rho'}.
\end{aligned}
$$

This gives

$$\left|F_i'^j\right| \le 16\left(\frac{2^5 A}{\sigma\rho}\right)^2 \mathrm{e}^{-|i-j|\rho'}.$$

2(b):

$$\begin{aligned}\sum_{i\in\mathbb{Z}}\left|F_i'^j\right| &\le 16\left(\frac{2^5 A}{\sigma\rho}\right)^2 \sum_{i\in\mathbb{Z}}\mathrm{e}^{-|i-j|\rho'} \\ &\le 16\left(\frac{2^5 A}{\sigma\rho}\right)^2 \sum_{i\in\mathbb{Z}}\mathrm{e}^{-|i|\rho'}\end{aligned}.$$

therefore

$$\sum_{i\in\mathbb{Z}}\left|F_i'^j\right| \le 16\left(\frac{2^5 A}{\sigma\rho}\right)^2 \frac{2}{1-\mathrm{e}^{-\rho'}} \tag{10}$$

It follows from the generalized Young inequality [12] (page 9) that (10) implies the desired estimate of F' in the operator norm on $\ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$.

3(a): Since $(D'-D)_i^i = F_i^i$ then $v_i' = v_i + F_i^i$ therefore $\partial_\theta v_i' = \partial_\theta v_i + \partial_\theta F_i^i$ thus

$$\left|\partial_\theta v_i'\right| \ge \left|\partial_\theta v_i\right| - \left|\partial_\theta F_i^i\right| \ge \alpha - \sqrt{\varepsilon} := \alpha'.$$

3(b): In order to estimate $\left|\partial_\theta F_i'^j\right|$, we have to find an upper bound of $\left|\partial_\theta X_i^j\right|$.

We have $\partial_\theta X_i^j = -\frac{\partial_\theta F_i^j}{v_i - v_j} + F_i^j \frac{\partial_\theta v_i - \partial_\theta v_j}{\left(v_i - v_j\right)^2}$ then

$$\begin{aligned}\left|\partial_\theta X_i^j\right| &\le \sqrt{\varepsilon}\frac{N^\tau}{\alpha\kappa} + \varepsilon\left(\frac{N^\tau}{\alpha\kappa}\right)^2 \left|\partial_\theta v_i - \partial_\theta v_j\right|, \\ &\le \sqrt{\varepsilon} C_1.\end{aligned}$$

which implies:

$(*)_1$

$$\begin{aligned}\left|\partial_\theta\left(\left[X, F^N\right]\right)_i^j\right| &= \left|\partial_\theta\left(\sum_{k\in\mathbb{Z}} X_i^k \left(F^N\right)_k^j - \left(F^N\right)_i^k X_k^j\right)\right| \\ &\le \sum_{k\in\mathbb{Z}} \left|\partial_\theta X_i^k\right| \left|\left(F^N\right)_k^j\right| + \left|X_i^k\right| \left|\partial_\theta\left(F^N\right)_k^j\right| + \left|\partial_\theta\left(F^N\right)_i^k\right| \left|X_k^j\right| \\ &\quad + \left|\left(F^N\right)_i^k\right| \left|\partial_\theta X_k^j\right| \\ &\le \sum_{k\in\mathbb{Z}} \sqrt{\varepsilon} C_1 \left|\left(F^N\right)_k^j\right| + \sqrt{\varepsilon}\left|X_i^k\right| + \sqrt{\varepsilon}\left|X_k^j\right| + \sqrt{\varepsilon} C_1 \left|\left(F^N\right)_i^k\right| \\ &\le \varepsilon^{3/2} M_1.\end{aligned}$$

In the same way we get:

$(*)_2$

$$\left|\partial_\theta\left(XDX\right)_i^j\right| \le \varepsilon^{3/2} M_2.$$

$(*)_3$

$$\left|\partial_\theta\left(XFX\right)_i^j\right| \le \varepsilon^{3/2} M_3.$$

$(*)_4$

$$\left(X^n\right)_i^j = \sum_{\ell_{n-1}\in\mathbb{Z}} \sum_{\ell_{n-2}\in\mathbb{Z}} \cdots \sum_{\ell_1\in\mathbb{Z}} X_i^{\ell_1} X_{\ell_1}^{\ell_2} \cdots X_{\ell_{n-1}}^j .$$

Since $\left|\left(X^k\right)_i^j\right|$ is bounded for all $k<n$ then $\left|\partial_\theta\left(X^n\right)_i^j\right| \le \varepsilon^{3/2} C_2$ therefore

$$\left|\partial_\theta\left(\sum_{\substack{m+n\ge 2\\(m,n)\ne(1,1)}} \frac{(-X)^n}{n!}\left(D+F^N\right)\frac{X^m}{m!}\right)_i^j\right| \le \varepsilon^{3/2} M_4$$

and

$$\left|\partial_\theta\left(\sum_{m+n\ge 0} \frac{(-X)^n}{n!}\tilde{F}_N\frac{X^m}{m!}\right)_i^j\right| \le \varepsilon^{3/2} M_5$$

where all constants M_i and C_i depend on N,α,κ and τ. It follows that

$$\left|\partial_\theta\left(F_i'^{\,j}\right)\right| \le 4\varepsilon^{3/2}\max\left(M_1,\cdots,M_5\right) \le \sqrt{\varepsilon'}.$$

3(c):

$$\left|\partial_\theta v'\right| \le \left|\partial_\theta v\right| + \sqrt{\varepsilon} \le c+\sqrt{\varepsilon}.$$

4) By construction of D' the result follows immediately. □

3. Induction

Let a, b such that $0 < a\tau + b < \frac{1}{4}$ and consider $\varepsilon_1=\varepsilon$, $\rho_1=\rho$, $\alpha_1=\alpha$, $A_1 = A$, $D^1=D$, $F^1=F$ and for all $n\ge 1$ we define the sequences

$$A_{n+1} = \frac{\left(N_{n+1}\right)^\tau}{\kappa\alpha_{n+1}}\varepsilon_{n+1} \qquad N_n = \frac{1}{\left(\varepsilon_n\right)^a \rho_n}$$

$$\varepsilon_{n+1} = 16\left(\frac{2^5 A_n}{\sigma_n\rho_n}\right)^2 \qquad \sigma_n = \left(\varepsilon_n\right)^b$$

$$\rho_{n+1} = \rho_n - \frac{\sigma_n\rho_n}{2} \qquad \alpha_{n+1} = \alpha_n - \sqrt{\varepsilon_n}$$

These parameters are defined in an iterative way and it is with which we will be able to define the matrices X_n, F^{n+1} and D^{n+1} satisfying

$$\mathrm{e}^{-X_n}\left(D^n+F^n\right)\mathrm{e}^{X_n} = D^{n+1}+F^{n+1} \tag{11}$$

where the matrices X_n, D^{n+1} and F^{n+1} are defined in the following way:

1) The matrix X_n is defined by $\begin{cases} \left(X_n\right)_i^j = 0 \ \text{ if } i=j \text{ or } |i-j|>N_n \\ \left(X_n\right)_i^j = -\dfrac{\left(F_n\right)_i^j}{v_i^n - v_j^n} \ \text{ otherwise} \end{cases}$ and satisfies the equation

$$\left[D^n, X_n\right] = \left(F^n\right)^{N_n} - D^{n+1} + D^n \tag{12}$$

where $\left(\left(F^n\right)^{N_n}\right)_i^j = \begin{cases} \left(F^n\right)_i^j & \text{if } |i-j| \le N_n \\ 0 & \text{otherwise} \end{cases}$.

2) $\left(D^{n+1} - D^n\right)_i^i = \left(F^n\right)_i^i$.

3) $F^{n+1}(\theta) = \mathrm{e}^{-X_n(\theta)}\left(D^n(\theta) + F^n(\theta)\right)\mathrm{e}^{X_n(\theta)} - D^{n+1}(\theta)$.

and satisfying the property $\mathcal{P}_n$ described in the following proposition.

Proposition 3. Let $n \in \mathbb{N}$. If $\forall m \le n$,

$$\varepsilon_m \le \left(\frac{\kappa\alpha_m\rho_m^{1+\tau}}{2^6}\right)^{\frac{1}{1-(a\tau+b)}}$$

then the following property $\mathcal{P}_n$ is holds.

$$\mathcal{P}_n \begin{cases} 1. \left|\left(X_n\right)_i^j\right| \le A_n \mathrm{e}^{-|i-j|\rho_n} \\ 2. \left\|X_n\right\| \le \dfrac{2A_n}{1-\mathrm{e}^{-\rho_n}} \\ 3. \left|\left(\mathrm{e}^{\pm X_n} - I\right)_i^j\right| \le \dfrac{2^7 A_n}{\sigma_n\rho_n}\mathrm{e}^{-|i-j|\rho_{n+1}} \\ 4. \left|\left(F^{n+1}\right)_i^j\right| \le \varepsilon_{n+1}\mathrm{e}^{-|i-j|\rho_{n+1}} \\ 5. \left\|F^{n+1}\right\| \le \dfrac{2\varepsilon_{n+1}}{1-\mathrm{e}^{-\rho_{n+1}}} \\ 6. \left|\partial_\theta v^{n+1}\right| \ge \alpha_{n+1} \\ 7. \left|\partial_\theta\left(F^{n+1}\right)_i^j\right| \le \sqrt{\varepsilon_{n+1}} \\ 8. \left|\partial_\theta v^{n+1}\right| < c_{n+1} = c_n + \sqrt{\varepsilon_n} \\ 9. \left|\left(D^{n+1} - D^n\right)_i^i\right| \le \varepsilon_n. \end{cases}$$

Proof. A direct application of Lemma 2 allows us to obtain the desired result for each *n*. □

4. Study of Convergence

Now we will deal with the study of the convergence of our iteration. We will therefore look at the conditions and the size of ε with which we will have the convergence, this will be the goal of the next lemma. Finally, we conclude with the proof of Theorem 1 which is a simple deduction of Proposition 3 and Lemma 4.

Lemma 4. Suppose that

$$\left(\frac{1}{2^9\alpha^3}\right)^{1-(a\tau+b)} < \alpha\kappa\rho^{\tau+1} < 2^{\tau+7}3$$

$$\left(2^6\alpha^2\right)^b \ge 2; \quad 0 < a\tau + b < \frac{1}{4}.$$

Then for $\varepsilon < \frac{1}{2^{37}}\left(\frac{\kappa\rho^{\tau+1}}{2^\tau 3}\right)^4 \alpha$ we have for all *n*:

1) $\varepsilon_n \le \left(\frac{1}{2^6\alpha^2}\right)^{(3/2)^n} \quad \forall \alpha \ge \frac{11}{2^4}$.

In particular $\lim_{n\to\infty} \varepsilon_n = 0$.

2) $\varepsilon_n \le \left(\frac{\kappa\alpha_n\rho_n^{\tau+1}}{2^6}\right)^{\frac{1}{1-(a\tau+b)}}$.

Proof: 1) The result is holds for $n=1$. Suppose that the result remain holds for $1,2,\cdots,n$ thus $\alpha_1 > \alpha_2 > \cdots > \frac{2}{3}\alpha$. Now we shall prove that the result is also true for $n+1$.

Let

$$M = \sum_{j=1}^{+\infty}\left(\frac{1}{2\left(1-(a\tau+b)\right)}\right)^j,$$

we have

$$\varepsilon_{n+1} = 16\left(\frac{2^5 A_n}{\sigma_n\rho_n}\right)^2 = 2^{14}\left(\frac{1}{\kappa\alpha_n\rho_n^{\tau+1}}\right)^2 \varepsilon_n^{2(1-(a\tau+b))} = \cdots$$

$$= \left[2^{14\sum_{k=1}^{n}\left(2(1-(a\tau+b))\right)^{-k}} \prod_{k=1}^{n}\left(\frac{1}{\kappa\alpha_k\rho_k^{\tau+1}}\right)^{2\left(2(1-(a\tau+b))\right)^{-k}} \varepsilon\right]^{\left(2(1-(a\tau+b))\right)^n}.$$

Since

$$\prod_{k=1}^{n}\left(\frac{1}{\kappa\alpha_k\rho_k^{\tau+1}}\right)^{2\left(2(1-(a\tau+b))\right)^{-k}} \le \left(\frac{1}{\kappa\alpha_n\rho_n^{\tau+1}}\right)^{2\sum_{k=1}^{n}\left(2(1-(a\tau+b))\right)^{-k}},$$

hence

$$\varepsilon_{n+1} \le [2^{14M}\left(\frac{1}{\kappa\alpha_n\rho_n^{\tau+1}}\right)^{2M}\varepsilon]^{\left(2(1-(a\tau+b))\right)^n}$$

$$\le \left[2^{28}\left(\frac{2^\tau 3}{\kappa\alpha\rho^{\tau+1}}\right)^4 \varepsilon\right]^{(3/2)^n},$$

which proves that $\lim_{n\to+\infty}\varepsilon_n = 0$.

2) By 1) we have $\forall n, \varepsilon_n \le 1$ then $\varepsilon_{n+1} = 2^2\left(\frac{2^6\varepsilon_n^{1-(a\tau+b)}}{\kappa\alpha_n\rho_n^{\tau+1}}\right)^2 \le 1$, thus $\frac{2^6\varepsilon_n^{1-(a\tau+b)}}{\kappa\alpha_n\rho_n^{\tau+1}} \le 1$ hence $\varepsilon_n \le \left(\frac{\kappa\alpha_n\rho_n^{\tau+1}}{2^6}\right)^{\frac{1}{1-(a\tau+b)}}$. □

Remark. 1) One can assume without loss of generality that $\alpha = 1$ and we have the same result, in fact the operators H_θ and αH_θ have the same spectral properties.

2) The real b exists and satisfying all conditions.

Proof of Theorem 1. 1) The operator H_θ is identified to matrix $D+F$ with $\left|F_i^j\right| \le (e\varepsilon)\mathrm{e}^{-|i-j|\rho}$. Then for $\rho = 1$ and $\varepsilon < \frac{\kappa\alpha\rho^{\tau+1}}{3e2^{\tau+37}}$ we have the existence of matrices X_n and D^{n+1} for all $n \in \mathbb{N}$ such that for all θ,

$$\left(e^{X_1(\theta)}\cdots e^{X_n(\theta)}\right)^*\left(D(\theta)+F(\theta)\right)e^{X_1(\theta)}\cdots e^{X_n(\theta)}=D^{n+1}(\theta)+F^{n+1}(\theta),$$

where $D^{n+1}(\theta)$ is a diagonal matrix, $\left\|F^{n+1}\right\|\le\varepsilon_{n+1}\frac{1}{1-e^{-\rho_{n+1}}}$,

$\left|\left(e^{\pm X_n}-I\right)_i^j\right|\le\frac{2^7 A_n}{\sigma_n\rho_n}e^{-|i-j|\rho_{n+1}}$ and $\left|\left(D^{n+1}-D^n\right)_i^i\right|\le\varepsilon_n$.

Therefore $F^n(\theta)\to 0$ and $D^n(\theta)\to D^\infty(\theta)$ with $D^\infty(\theta)$ is a diagonal matrix. All convergence are fulfilled for all θ.

On the other hand $e^{X_1(\theta)}\cdots e^{X_n(\theta)}\to U(\theta)$ in norm and for all θ with $U(\theta)$ is an orthogonal matrix. In fact: Let $U_j(\theta)=e^{X_j(\theta)}$ we have $\prod_{j\ge1}U_j(\theta)$ converges if and only if $\sum_{j\ge1}\left\|U_j(\theta)-I\right\|$ converges, now since $\left\|U_j(\theta)-I\right\|\le\frac{2\sqrt{\varepsilon_j}}{1-e^{-\rho_{j+1}}}$ then we have the existence of U for all θ. Moreover from Lemma 4 and for $\varepsilon_0=\frac{1}{e2^{37}}\left(\frac{\kappa}{2^\tau}3\right)^4\alpha$ the matrix $D^\infty(\theta)$ is pure point with finite-dimensional eigenvectors for all θ and the measure of $\sigma(D)\setminus\sigma(D+F)$ goes to 0 as $\varepsilon\to 0$. The eigenvectors of $D+F$ are formed by the columns of U.

2) Let $\psi_\theta(t,n)=e^{-itH_\theta}\psi$ for $\psi\in\ell^2(\mathbb{Z})$, so by the exponential decaying of eigenfunctions we can easily deduce that

$$\sup_t\sum_{n\in\mathbb{Z}}\left(1+n^2\right)\left|\psi_\theta(t,n)\right|^2<\infty.$$

$\square$

5. Conclusion and Suggestion

The result in Theorem 1 improves the previous one by Eliasson, such that dynamical localization is proven with an appropriate potential. To my knowledge, there are no results on the spectral properties of Schrödinger operators with discontinuous potential. The ideas presented in this paper can be used to obtain new results for several models with discontinuities. For example, one can consider the quasi-periodic operator defined for θ in the torus $\mathbb{T}$ with a piecewise smooth potential or even functions in a piecewise Gevrey class, and see if similar localization results can be obtained.

Conflicts of Interest

The authors declare no conflicts of interest.

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