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Examining Students' Concept Images in Mathematics: The Case of Undergraduate Calculus

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Abstract

A mathematical concept's mental representation encompasses the collection of mental images gradually formed by a learner as they develop a cognitive grasp of the concept. This mental representation is known in literature as concept image. It might not be completely accurate and could encompass certain aspects of the concept's formal definition. A sound understanding of students' concept images of topics in mathematics and how those images can affect their cognitive development would be invaluable for teachers when instructing learners. Researchers in mathematics education have long attempted to comprehend the formation of concept images in students' minds pertaining to numerous topics and the evolution of those concept images over time. Therefore, this paper provides guidelines that help in developing a concept image, which contributes to positive learning outcomes. It also includes a brief discussion on the utilization of technology in shaping the concept image.

Keywords

Concept Image, Concept Definition, Calculus, Mathematical Concepts, Integrating

1. Introduction

Almost all mathematical concepts have a formal definition. However, students may not necessarily use the precise mathematical definition to understand the underlying notion or apply it in problem solving. Instead, they may find it more convenient to use what is known as a "concept image" which is comprised of all mental pictures formed in the mind with regard to the concept and the properties characterizing it. Tall & Vinner (1981) describe the concept of "concept images" in their paper as follows.

> "The concept image consists of all the cognitive structure in the individual's mind that is associated with a given concept. This may not be globally coherent and may have aspects which are quite different from the formal concept definition." (p. 151)

So, a concept image is composed of the set of all mental images or conceptions that a student develops in mind regarding a mathematical concept. These mental pictures can be of any kind, chart, picture, graph, diagram, formula, or some other symbolic form. For example, a student may interpret a function as a rule, a graph, an operation, a table of values, a relationship between two sets, a mapping from one set to another, or some other kind of a representation. Tall & Vinner (1981) refer to a distinction between the formal definition of mathematical concepts and the cognitive processes used to conceive those concepts. This is essentially the conflict between a formal mathematical definition and the concept image stemming from different aspects of the formal definition.

There are some other properties that are used to characterize students' concept images in mathematics. As such, a concept image may not necessarily be restricted to a single symbol, formula, or a mental picture, but it is composed of the "total cognitive structure" involving all the mental processes consciously and unconsciously shaping the meaning and usage of the concept (Tall & Vinner, 1981). Also, a concept image is not a mental picture/s built all at once, rather something that is developed over time as the student gets exposed to the concept repeatedly and it could go through revisions and modifications as the student progresses in their learning journey. This means that the concept image is always "work in progress" and dynamic in nature, hence some authors including Bingolbali & Monaghan (2008) use the descriptor, "developing" in describing concept images.

The potential problem with a concept image is that the set of mathematical objects considered by the student to be examples of a concept image may not necessarily be the set of objects that comply with the formal definition pertaining to that concept (Tall & Vinner, 1981; Vinner & Dreyfus, 1989). Tall & Vinner also claim that some or most steps in the development of the concept image may not necessarily be rational or consistent because different stimuli can influence the concept image differently over time, especially as the student's understanding of the content matures with regular exposure to more advanced material. Furthermore, Tall & Vinner (1981) state that this difference may lead to a cognitive conflict at a later stage of the process, particularly when conflicting concept images are evoked simultaneously.

Interestingly, concept images constitute a vital component in understanding students' mathematical comprehension of concepts, especially for teachers in planning their pedagogical approaches. So, teachers and researchers in mathematics education have attempted to explore how students build concept images pertaining to various mathematical concept and how those concept images deviate from the exact definition, and how they affect (positively or negatively) in the students' acquisition and retention of mathematical knowledge. This project makes an effort to analyze some of the prominent research in recent literature that study the case of undergraduate calculus topics, especially functions, limits, differentiation, integration, and differential equations. The aim of this paper is to provide a comprehensive overview of the concept images in calculus, and contribute to the ongoing discourse on how to improve students' understanding of this fundamental mathematical topic.

2. Student's Concept Image in Mathematics

Tall & Vinner (1981) discuss extensively the development of undergraduate students' concept images pertaining to the topics of limits and continuity of functions which is the basic building block of calculus. Individual concept images can be different from the formal theory or precise mathematical definitions and may be characterized by some aspects which are likely to lead to cognitive confusion (Tall & Vinner, 1981). The portion of the concept image pertaining to a certain concept evoked at a particular point of time may depend on the context under consideration. That is, a concept image for a single concept may have different facets which are evoked in accordance with the features of the problem at hand (Tall & Vinner, 1981).

However, it is only in cases where conflicting aspects of the concept image are invoked simultaneously that there may arise a noticeable sense of conflict. The unease subconsciously caused by this conflict or confusion is the reason for the unease experienced by individuals in problem solving (Tall & Vinner, 1981). While different parts of the concept image may not necessarily be consistent, another potential conflict factor that Tall & Vinner (1981) warn about is when the concept image is at variance with the formal definition of the concept.

The inability to formulate an accurate concept image and the negative consequences of having a flawed concept image (which has conflicting aspects not just with the formal definition but also with other aspects of the same image), is likely to significantly disturb a student's understanding of a mathematical concept (Tall & Vinner, 1981; Vinner & Dreyfus, 1989).

Many authors have investigated possible approaches to enhance or enrich students' concept images in undergraduate calculus topics. Tall & Al (1987) has examined how a concept image could be improved using technology besides the traditional methods of teaching. When a learner is exposed to a new mathematical concept, they produce an idiosyncratic concept image in mind with implicit properties pertaining to the context, but this might lead to cognitive conflict at a later phase of the learning process (Tall & Al, 1987). Some research has found that the incorporation of technology can help build a more coherent and wholesome concept image with flexibility to move between representations and transferability to new concepts (Tall & Al, 1987). Technology can be used to emphasize the key properties of a new concept by manipulating examples and non-examples in ways that are moderately complex (Tall & Al, 1987).

Some early researchers have also attempted to explore how these concept images differ from the formal definition of a function. Vinner & Dreyfus (1989) conduct a comprehensive series of interviews involving 271 college students and 36 junior high school teachers in order to analyze their concept image for the concept of the function definition. Vinner & Dreyfus (1989) use a questionnaire with problems that capture the cognitive schemas for the concept of a function that are activated in two types of problems, identification and construction with the objective of testing the participants' understanding of the concept of functions at a deeper level. The findings of Vinner & Dreyfus (1989) conclude that most of the definitions and images provided were of primitive nature, except for those made by teachers and mathematics majors.

Vinner & Dreyfus (1989) also emphasize the importance of the implications for teaching that can be derived from a study of common concept images of the concept of functions that college students have. Another significant finding of Vinner & Dreyfus (1989) is that the course level of the students is also a good indication of the accuracy or appropriateness of their concept images, the higher the course level of the students, the fewer cases of compartmentalization.

Some authors also argue that a concept image is not formed at once in just one step, instead it occurs in several stages. It is also important to note that the intermediate steps in the process of developing a concept image may exhibit some inconsistencies and conflicting aspects (Tall & Vinner, 1981; Thompson, 1994). While Tall & Vinner (1981) regard a concept image as the blending of mental pictures into categories of conventional mathematical vocabulary, Thompson (1994) attempts to focus more on the dynamics of mental operations pertaining to a concept image. Thompson (1994) also emphasizes that these two notions of concept image are not necessarily inconsistent, but just have some different foci. Thompson (1994) proclaims that students experience considerable issues with understanding the fundamental theorem of calculus under concern and most of these difficulties stem from poor concept images of rate of change and inadequately as well as inappropriately developed images of functional co-variation. Thompson (1994) finds out that notational opacity may also hinder the students' concept images, especially when drawing connections between and among concepts.

Giraldo et al. (2003) explore, from a pedagogical perspective, the inherent limitations of computational descriptions pertaining to mathematical concepts with an emphasis on the concept of derivative. Interestingly and ironically, they claim that such limitations can, in a subtle way, be used to enrich students' concept images. This is in contrast to what Tall & Vinner (1981) suggest about conflicting aspects of a concept image. They analyze the student's mental attitude towards conflict situations and suggest that conflicts contribute to enriching the concept of derivative and related ideas, particularly because a conflicting problem encourages the student to dive into deeper reasoning. However, they warn that this is not always the case as some students do not have the ability to grapple with conflicts. Giraldo et al. (2003) allude to a "rich" concept image as a cognitive structure which includes the formal definition as well as a combination of many "cognitive units" which are linked together, and the term, "cognitive units" is used to refer to any symbols or other representations associated with the concept under concern.

The discussion of concept image and concept definition, even though forms a crucial construct in mathematics education, has been restricted to cognitive studies (Bingolbali & Monaghan, 2008). So, interestingly, Bingolbali & Monaghan (2008) investigate undergraduate students' concept images in their article in the context of learning by focusing on the influence that teaching and departmental affiliation have on students' developing concept images. Bingolbali & Monaghan (2008) argue that the students from mathematics major had a concept image of the derivative inclined towards the tangent aspects whereas that of mechanical engineering students was more drawn towards the rate of change aspects. Another important aspect highlighted by Bingolbali & Monaghan (2008) is that students' developing concept images and how they draw connections between different forms are closely associated with teaching practices and common departmental perspectives. They argue that it is interesting to explore the different aspects of the concept images associated with the same concept definition resulting from departmental influences. So, Bingolbali & Monaghan (2008) make a unique contribution to the literature in the sense that it explores how the departmental affiliations influence the nature of students' concept images in calculus topics.

Some researchers have attempted to explore how the concept image of a prerequisite topic affects the concept image of a subsequent topic. Vincent (2016) analyze how students' concept images of tangent lines affect their understanding of the concept of derivative. They argue that a richer concept image of tangent lines as a dynamic limiting process of the secant lines to the graph at neighboring points enriches students' understanding of the concept of derivative. Vincent (2016) also highlight the fact that the definition of tangency should be emphasized explicitly as the limiting position of secant lines, while it will still be useful to use imprecise definitions such as the line tangential to the graph at a point or line with a slope equal to the derivative of the graph at the point. The consistent use of "shortcut" or inexact definitions may cause to create pseudo-objects (an object which appears to be part of the concept image, but actually is not) within students' concept images pertaining to the concepts of tangent line and derivative, as such pseudo-objects may carry over to the students' concept image of derivatives and also for structural understandings (Vincent, 2016).

Some recent researchers have used qualitative descriptive approaches to categorize and analyze students' concept images of some calculus concepts and the influence that concept images have on the mental process of solving conceptual problems in calculus. Nurwahyu & Tinungki (2020) broadly categorize students' concept images in calculus topics into four types, namely incipient concept image, instrumental concept image, relational concept image, and formal concept image. Nurwahyu & Tinungki (2020) conclude that the most widely held type of concept images is the instrumental concept images and the least common one is the formal concept images. This result is consistent with the results of the experiment conducted by Vinner & Dreyfus (1989).

Nurwahyu & Tinungki (2020) further conclude that the students' motivation for solving specific types of problems is enhanced by the type of concept image they have pertaining to the relevant concepts in calculus. It is interesting to see how the authors have explored the relationship between concept images and the students' approach to problem solving and motivation for problem solving. This aspect makes the scope and subject of this paper somewhat different from the rest of the papers analyzed here. Teachers need to always check for the students' concept images and improve any rough edges as the concept images heavily impact students' motivation and problem-solving skills (Nurwahyu & Tinungki, 2020).

Some researchers have explored different ways in which concept images can be taught to students. Nair (2010) suggests that multiple perspectives should be used in teaching concepts, this should include at least analytic, graphic, and numeric forms of concepts. Nair (2010) further reinforces that the visual impact of graphical representations could help avoid conflicts causing uncontrollable concept images. In mathematics, a student cannot smoothly progress through a course or a sequence of courses unless they acquire and retain a solid understanding of each topic in the course/s. So, students their concept images of previous topics in order to construct concept images of new topics. In other words, concept images from one topic are prerequisite for forming concept images in subsequent topics. Juter (2007) concludes that students need to develop their concept images both in depth and width along with abstraction skills and strong links between and among numerous topics. Tall & Vinner (1981) brings forth a rather controversial but realistic point that students could still perform well in exams even with erroneous concept images:

> "We have to admit that in applications or in regular achievement tests the concept image does not have to play a crucial role. Students can succeed in examinations even when having a wrong concept image. Hence if we do not care about the concept image we do not have to bother with it. On the other hand, if we do care about it then what we do is not enough."

Some researchers have explored intuitive and constructive ways in which students can be exposed to mathematical concepts. Jones (2015) investigates how the multiplicatively-based summation concept (the symbolic form of adding up pieces) is more productive in helping students develop an intuitive understanding of a variety of definite integrals than the conceptualizations of antiderivative and are under the curve. What makes the multiplicatively-based summation conception easier for students to grasp is the ease of constructing a concept image involving a representative rectangle to understand the multiplicative interaction between the integrand and the differential.

It is also interesting to see that some researchers use frameworks to explore students' concept images in relation to mathematical definitions. Raychaudhuri (2008) explores students' concept images of the solution to a first-order DE using the framework of context-entity-process-object (CEPO) which befits dynamical definitions in mathematics. A dynamic definition in mathematics is a definition is one where the object being defined is not explicit in the definition. In the CEPO framework, context of the problem distinguishes between algebraic equation vs differential equation; solution as an entity is a function; solution as an object is solution as an entity with the process mandating the requisite property; process usually involves dual processes, namely generating process and defining process.

There has been some increasing interest in research that explores the importance of work inspired by Realistic Mathematic Education (RME) for enhancing students' conceptual understanding and enriching their concept images. This innovative approach derives its inspiration from the recent drive among educators towards extending students' intuitive and informal ways of reasoning to construct solid and formal ways of reasoning. Rasmussen & Blumenfeld (2007) experiment with an RME-inspired method to enrich students' conceptual understanding of topics in differential equations. They analyze a scenario where the students reinvent an approach to determine the solution to a system of two simultaneous ODEs with a novel "eigenvector first approach" as opposed to the conventional "eigenvalue first approach". They use this example in order to emphasize the importance of designing activities that underpin the model-of/model-for transition. This transition intends to build intuition through concrete examples in order to construct formal mathematical theory thus enhancing students' intellectual growth.

3. Guideline for effectively Constructing Concept Image in Calculus

To construct an effective concept image in calculus, it is essential to follow certain guidelines. First, it is important to start with a clear definition of the concept and its key components which must have been fully understood by learners. This provides a foundation for understanding and organizing the information. Next, it is crucial to identify and understand any relevant properties or rules associated with the concept. This helps in forming connections and applying the concept in different contexts.

It is beneficial to visualize the concept using diagrams or graphs whenever possible. This helps in creating a concrete representation of the concept, making it easier to understand and remember. It is important to practice applying the concept in problem-solving scenarios. This reinforces understanding and helps in developing a deeper grasp of the concept.

Constructing a clear and effective concept image in calculus requires attention to both structure and synthesis. By organizing the information and synthesizing different ideas, we can form a comprehensive mental representation of the concept. Following guidelines such as starting with a clear definition, identifying relevant properties, visualizing concepts, and practicing problem-solving can aid in effectively constructing a concept image in calculus.

3.1. Structure and Synthesis

When studying calculus, it is essential to develop a strong understanding of the concepts involved. One way to enhance this understanding is by constructing a clear and effective concept image. To construct a concept image in calculus, it is crucial to pay attention to the pattern and structure of every piece of information that form the concepts (Warren, 2005; Mulligan & Mitchelmore, 2009). These pieces can then be synthesized and summarized.

The structure of the concept image involves organizing the information in a logical and coherent manner. This means identifying the key elements and relationships within the concept. By understanding the structure, we can better comprehend how different components fit together and interact. For example, in calculus, understanding the structure of a derivative involves recognizing how it is calculated using limits and how it relates to the original function. By organizing this information, we can form a clear mental picture of the concept.

Synthesis, on the other hand, involves integrating different pieces of information to form a complete understanding. It requires connecting various ideas and applying them to solve problems or analyze situations. In calculus, synthesis could involve combining the concept of limits with that of continuity to understand differentiability. In multivariable calculus, the concept of partial derivatives can be combined with the notion of continuity to arrive at differentiability. By synthesizing these ideas, we can form a comprehensive concept image that allows us to apply calculus principles effectively.

Structure and synthesis are integral aspects of constructing a robust concept image in calculus. A well-organized mental framework (structure) allows students to navigate complex calculus concepts efficiently, see hierarchical relationships, and visualize geometric interpretations (Klang et al., 2021). It aids problem-solving by helping students identify relevant information and apply appropriate techniques. It enhances retention, transfer of knowledge to new scenarios, and effective communication of mathematical ideas (Bressoud et al., 2016). Synthesis, the ability to connect related concepts, enables students to bridge gaps between different topics in calculus, fostering a deeper understanding and facilitating adaptability in problem-solving. These skills also promote metacognition, empowering students to monitor their learning and make informed decisions about improving their calculus proficiency. Structure and synthesis are pivotal for mastering calculus and succeeding in advanced mathematics and related disciplines.

3.2. Keywords, Labelling, and Inter-Relationships

In order to effectively construct a concept image in calculus, it is important to understand the role of keywords, labeling, and inter-relationships. Keywords are essential in defining and conveying the main ideas and concepts within a mathematical problem (Hegarty et al., 1995). They provide a framework for understanding and organizing the information presented (Campbell et al., 2007, Powell et al., 2022). When constructing a concept image, it is important to identify and highlight the main keywords that are relevant to the problem at hand. By doing so, one can focus on the essential elements and better understand the problem.

Labeling is another crucial aspect of constructing a concept image in calculus. By assigning labels to different variables, functions, and equations, it becomes easier to visualize and comprehend the relationships between different mathematical entities. Labeling allows for clear communication and understanding of the problem, as it provides a visual representation of the concepts being discussed. Additionally, labeling helps in organizing information and facilitates problem-solving by providing a structure for analysis.

Inter-relationships between different mathematical concepts play a vital role in constructing a concept image in calculus. Understanding how different variables and functions interact with each other is essential in grasping the underlying principles of calculus. By identifying and analyzing these inter-relationships, one can develop a deeper understanding of the concepts being studied. This understanding enables effective problem-solving and allows for the application of calculus principles in various scenarios.

To effectively construct a concept image in calculus, paying attention to keywords, labeling, and inter-relationships is crucial. These guidelines provide a framework for organizing and comprehending mathematical problems. By highlighting keywords, assigning labels, and analyzing inter-relationships, one can develop a clear and concise concept image that aids in problem-solving and understanding complex calculus concepts. For instance, when studying limits of functions, an important keyword is "approach". These guidelines enhance mathematical thinking and promote a deeper understanding of calculus.

3.3. Visual Representation

In the field of mathematics, visual representation plays a crucial role in understanding complex concepts and problem-solving techniques (Presmeg, 2006; Presmeg, 2020). Visual representations, such as graphs and diagrams, provide a clear and concise way to convey mathematical ideas. Graphs and diagrams should be labeled appropriately and should clearly represent the mathematical concepts being discussed. For example, when depicting the relationship between a function and its derivative, it is important to label the axes, include key points on the graph, and provide a clear indication of the slope at each point.

Another important guideline is to use color and shading effectively. Color can be used to highlight important features or to distinguish between different elements in a visual representation. For example, in a graph showing the area under a curve, the shaded region can be colored differently to draw attention to this particular area of interest.

Furthermore, it is important to consider the scale and proportion when constructing a concept image in calculus. The size of the visual elements should be appropriate for the information being conveyed. For example, if a graph represents data over a large range of values, it may be necessary to use a logarithmic scale to ensure that all relevant information is visible.

The relationship between student visual representation and student concept image in calculus is a crucial aspect of mathematical learning. Visual representations, such as graphs and diagrams, serve as powerful tools for students to develop and communicate their understanding of mathematical concepts (Pape & Tchoshanov, 2001). The concept image, on the other hand, represents how a student mentally constructs and comprehends these mathematical ideas (Rösken & Rolka, 2007). When students effectively connect their visual representations with their concept image, it promotes a deeper and more meaningful understanding of calculus concepts, facilitating problem-solving and mathematical reasoning, ultimately enhancing their overall mathematical proficiency.

Overall, constructing an effective concept image in calculus requires careful consideration of visual elements, color usage, and scale. By following these guidelines, educators can enhance students' understanding of complex mathematical concepts and facilitate their problem-solving abilities. Visual representations are common in mathematics, and by carefully exploring this fact, students can develop a strong conceptual understanding of calculus.

4. Predicting Learning Outcomes in Calculus through Concept Image

The main goal of pedagogy is to improve the learning outcomes of students. One approach to achieving this improvement is through the concept image, which refers to a mental representation of a mathematical concept. By understanding students' concept images, educators can gain insights into students' learning journey and understanding of calculus. This enables teachers to identify challenges of students while learning and consequently provide viable solutions with huge impact on learning outcomes.

Undergraduate calculus is a foundational course that lays the groundwork for further studies in mathematics and related fields. In many universities across the world, calculus topics are compulsory for STEM students. This is mainly because of the diverse applications of calculus to different areas of STEM. Therefore, it is crucial to identify and address any misconceptions or gaps in students' understanding early on. By examining their concept images, educators can identify areas where students may struggle and tailor their instruction accordingly. In a recent study Siagian et al. (2021) conducted to investigate the concept image of students about variables in mathematics, it was discovered that a thorough grasp of the variable concept in algebra has an impact on students' capacity to accurately formulate algebraic expressions, and consequently leads to improved performance in the subject.

The concept image can be assessed through various methods, such as interviews, surveys, or written tasks. These assessments provide valuable information about how students perceive and understand calculus concepts. For example, if a student has a strong concept image of derivatives as rates of change, they are more likely to succeed in applying derivative rules and solving related problems. On the other hand, if a student's concept image of derivatives is limited to memorizing formulas without understanding their meaning, they may struggle with more complex calculus concepts.

By understanding how concept image influences learning outcomes, educators can design targeted interventions to support struggling students and enhance their understanding of calculus. This could involve providing additional resources, offering one-on-one tutoring, or implementing interactive teaching methods that promote conceptual understanding. By addressing misconceptions and strengthening students' concept images, educators can improve learning outcomes and set students up for success in future mathematics courses (Ertekin et al., 2014).

5. Technology in Concept Image: Implications and Complications for Calculus

In his article, Abramovich (2014) aptly observed and argued that the advent of sophisticated computer programs, such as Wolfram Alpha, has rendered various problems within the secondary mathematics curriculum somewhat obsolete, as these programs readily solve them. This assertion is applicable to advanced calculus topics at the undergraduate level. Rather than entertaining the notion of eradication of technological tools from the pedagogy of mathematics, educators can diligently explore methodologies for seamlessly integrating technology into the teaching of calculus within the modern academic setting. Resources such as Symbolab, Wolfram Alpha, Desmond, and GeoGebra prove to be exceedingly advantageous in the comprehension and study of multivariate calculus.

The integration of technology into the concept image of calculus has both implications and complications for undergraduate students. Technology has the potential to enhance the understanding and visualization of complex mathematical concepts in calculus. With the use of graphing calculators and computer software, students can explore graphs, analyze functions, and solve problems with greater efficiency and accuracy. This allows them to develop a deeper understanding of calculus concepts, such as limits, derivatives, and integrals. Additionally, technology can provide real-world applications of calculus, making the subject more relevant and engaging for students.

The inclusion of technology in the concept image of calculus also presents some complications. While technology can be a valuable tool, it should not be used as a substitute for understanding the underlying mathematical principles. Students may become overly reliant on calculators and software without fully grasping the concepts behind the calculations. This can lead to a shallow understanding of calculus and hinder their ability to apply it in problem-solving contexts. Furthermore, there is a risk that students may misuse or misinterpret the results provided by technology, leading to errors in their calculations or conclusions.

To address these issues, it is important for educators to strike a balance between using technology as a learning tool and ensuring that students have a solid foundation in the fundamental concepts of calculus. This can be achieved by incorporating technology into classroom activities and assignments, while also providing opportunities for students to work on hand calculations and problem-solving without relying solely on technology. Additionally, educators should emphasize the importance of critical thinking and understanding the underlying mathematical concepts, rather than simply relying on technology to provide answers. Furthermore, educators should also consider formulating calculus questions in manners that stimulate critical thinking and discourage excessive reliance on technology without a complete grasp of concepts (Seaton & Tacy, 2022; Ashford 2021; Ertekin et al., 2014).

6. Conclusion

We conclude through the review of relevant literature that a concept image is the set of all mental structures that a student builds with respect to a mathematical concept as opposed to its formal definition. Research indicates that students use their concept images more often than the precise definition in problem solving activities. However, it is also emphasized that most concept images are at least partially flawed, and this could have a significant impact on their approach and ability to solve problems accurately. So, it is very important for teachers to have an idea of the common concept images that students have pertaining to different topics, the widespread issues with such concept images, and effective remedial issues to be taken.

It is interesting to note that there is an extensive body of literature made with reference to undergraduate students' concept images in topics of calculus. Despite the amazing body of work pertaining to undergraduate students' concept images in calculus topics, there seems to be a considerable lack of research underpinning the pedagogical approaches that aim to enhance students' concept images. While Rasmussen & Blumenfeld (2007) explore the use of innovative emergent models for enhancing students' conceptions in differential equations, it is expected that research in the future could explore more about different ways in which students' concept images can be improved. The use of technology and inquiry-based learning seems to have considerable potential in this respect.

Another facet of concept images that could be investigated by future researchers is using continuous assessments in ways that aim to improve concept images and rectify any rough edges in existing concept images. It would also be interesting to explore how evolution of students' developing concept images throughout a semester as they get exposed to a sequence of treated topics in a course. Moreover, another direction for future research would be to investigate the influence of students' concept images from different mathematical subjects on each other, for example, whether students' concept images on topics in linear algebra have an impact on those in calculus.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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