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Research Article

The Degree-Based Topological Indices for Two Special Families of Graphs of Diameter Three

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In this research article, we determine some vertex degree-based topological indices or descriptors of two families of graphs, i.e., $G = C_4(K_n)$ and $G = C_4(K_n) + v_1 v_3$, where $C_4(K_n)$ is a graph obtained by identifying one of the vertices of K_n with one vertex of C_4. Similarly, a graph formed by joining one of the vertices of K_n with one vertex of $C_4 + v_1 v_3$ is known as the $C_4(K_n) + v_1 v_3$ graph.

1. Introduction

Around the center of a century ago, theoretical experts found some interesting relationships between different properties of organic substances and those of molecular structure by analyzing a few invariants of the underlying molecular graph. These graph invariants are useful for molecular objects and are named as topological indices or topological descriptors. These are numerical parameters of graphs which can be used to describe the physiochemical properties like boiling point, molecular weight, density and refractive index of a molecule [1, 2]. The three main categories of topological indices are distance-based topological indices, degree-based indices and spectrum-based topological indices. Among these, degree-based topological indices are of great significance [1, 3, 4]. Some vertex degree-based topological indices are Randic or connectivity index, first and second Zagreb indices, Narumi Katayama and multiplicative Zagreb indices, atom bond connectivity index, augmented Zagreb index, geometric arithmetic index, harmonic index, and sum connectivity index [5]. The topological descriptors are beneficial tools to chemists that are provided by graph theory. The use of these theoretical invariants in QSPR/QSPR modeling has been a popular work in the past years [4, 6].

The first most widely used topological index in chemistry is the Wiener index. Initially, it was presented by the American Chemist Harold Wiener in 1947 and is denoted by $W(G)$. After its success, many other topological indices have been introduced that are based on Wiener's work [3, 7].

Milan Randic introduced the Randic index in 1975. It was the most popular index among all topological indices [2, 8]. The mathematical form of this descriptor is

$$R(G) = \sum_{\mu\sim\nu} \frac{1}{\sqrt{d_\mu(G) d_\nu(G)}}, \tag{1}$$

with a summation going with overall pairs of adjacent vertices of the graph G.

Later, mathematicians did not pay any attention to this index for almost two decades. However, Erdos together with Bollobas worked on this index and discovered the mathematical hidden in it. In 1998, they published their first paper on this index. When the mathematical community began to understand the importance of this descriptor, they started to do researches on it. There have been hundreds of articles and several books dedicated to this structural descriptor [9]. The chemists [1] found out some expressions during the study of molecular structure dependency of total pi-electron energy which includes the terms of the form

$$\begin{aligned} M_1(G) &= \sum_{\mu} d_\mu(G)^2, \\ M_2(G) &= \sum_{\mu\sim\nu} d_\mu(G) d_\nu(G). \end{aligned} \tag{2}$$

Immediately, it was identified that these two terms provide quantitative measures of the branching of the molecular carbon atom skeleton. In chemical theory, M_1 and M_2 are called first and second Zegreb indices. These two indices have been introduced by Gutman and Trinajestic in 1972. Initially, the first Zagreb index was also known as the Gutman index. But, Balaban et al. did not want to introduce these indices by the names of the discovers. After 1982, Balaban et al. included these two terms in the topological indices and named them as Zagreb indices. In 2012, multiplicative versions of multiplicative Zagreb indices have been proposed by Todeshine et al. and are known as Narum Katayama indices. They were the first topological index defined by the product of some graph theoretical quantities. In 1984, Narumi and Katayama determined the term $\prod_\mu d_\mu(G)^2$ and named it as the first multiplicative Zagreb index. It is denoted by $\prod_1(G)$. The second multiplicative Zagreb index and modified first multiplicative Zagreb index are defined, respectively, by

$$\begin{aligned} \prod_2(G) &= \prod_{\mu\sim\nu} d_\mu(G) d_\nu(G), \\ \prod_1^*(G) &= \prod_{\mu\sim\nu} \left[d_\mu(G) + d_\nu(G)\right], \end{aligned} \tag{3}$$

with a product going with overall adjacent vertices of a graph G [10, 11]. The first, second, and modified multiplicative Zagreb indices are known as Narumi Katayama multiplicative Zagreb indices. The geometric arithmetic index is the modified form of the Randic index. This topological descriptor was introduced by Vukicevic and Furtula in 2009. The mathematical form [12] of this index is

$$GA(G) = \sum_{\mu\sim\nu} \frac{\sqrt{d_\mu(G) d_\nu(G)}}{\left(d_\mu(G) + d_\nu(G)\right)/2}. \tag{4}$$

Another vertex degree-based topological index that keeps the spirit of Randic index is the atom bond connectivity index. This topological descriptor was introduced by Estrada et al. in 1998. This index has turned out to be an important index for the stability of Alkanes. The mathematical form [8, 13] of this index is

$$\mathrm{ABC}(G) = \sum_{\mu\sim\nu} \sqrt{\frac{d_\mu(G) + d_\nu(G) - 2}{d_\mu(G) d_\nu(G)}}. \tag{5}$$

Furtula et al. introduced the modified form of atom bond connectivity index in 2009 and named it as the augmented Zagreb index. The mathematical form [8, 14] of this descriptor is

$$\mathrm{AZI}(G) = \sum_{\mu\sim\nu} \left[\frac{d_\mu(G) d_\nu(G)}{d_\mu(G) + d_\nu(G) - 2}\right]^3. \tag{6}$$

In the investigation of heat creation in heptanes and octanes, the prediction capability of the augmented Zagreb index outperforms that of the atom bond connectivity index. Ali et al. investigated the correlation abilities of 20 vertex degree-based topological descriptors for the case of normal heats of formation and ordinary boiling factor of octane isomers, finding that the AZI produces the best results [15].

In 1980, Fajtlowiez introduced another topological index. In 2012, Zhang worked on this index and named it as the harmonic index. This index is defined as

$$H(G) = \sum_{\mu\sim\nu} \frac{2}{d_\mu(G) + d_\nu(G)}. \tag{7}$$

No chemical applications of this index have been found so far, but in the last few years, this index has attracted the great attention of theoreticians [16, 17]. Another vertex degree-based topological index was proposed by Zhou and Trinajstic and was named as the sum connectivity index. They observed in the definition of Randic index that the term $d_\mu \times d_\nu$ can be replaced by $\mathrm{d}_\mu + d_\nu$ which is defined by ([2, 18, 19]).

$$\mathrm{SCI}(G) = \sum_{\mu\sim\nu} \frac{1}{\sqrt{d_\mu(G) + d_\nu(G)}}. \tag{8}$$

Many topological indices are bond-additive, i.e., they can be presented as a sum of edge contributions and have the following form:

$$\sum_{\mu\nu\in E(G)} f(g(\mu), g(\nu)), \tag{9}$$

where $g(\mu)$ are usually degrees or the sum of distances from μ to all other vertices of G. Inspired by the most successful indices of this type, such as the Randic index, the second Zagreb index and others, a whole family of Adriatic indices [20] was defined. A particularly intriguing subclass of these descriptors is made up of 148 discrete Adriatic indices. They were examined on the testing sets provided by the International Academy of Mathematical Chemistry, and it had been shown that they have good predictive properties in many cases and it was found that they have good predictive properties in many cases. One of these useful discrete Adriatic indices is the symmetric division deg (SDD) index, which is defined as

$$\mathrm{SDD}(G) = \sum_{\mu\nu\in E(G)} \left(\frac{d_\mu}{d_\nu} + \frac{d_\nu}{d_\mu}\right), \tag{10}$$

where $E(G)$ is the edge set of a molecular graph G (a graph in which vertices correspond to atoms of a (hydrogen-suppressed) molecule and edges correspond to bonds between the atoms) and d_μ and d_ν denote the degrees of the vertices $\mu\nu \in V(G)$, respectively [21].

In this paper, we compute the degree-based topological indices, namely, Randic or connectivity index, Zagreb indices, Narumi-Katayama and multiplicative Zagreb indices, atom bond connectivity index, augmented Zagreb index, geometric arithmetic index, harmonic index, and sum connectivity index for two special families of graphs of diameter three. These graphs are undirected having no loops and multiple edges. In Section 2, the topological indices of families of graphs $C_4(K_n)$ are determined, while in Section 3, the topological indices of families of graphs $C_4(K_n) + v_1v_3$ are computed.

2. Topological Indices of Families of Graphs $C_4(K_n)$

In this section, we have determined and computed the closed formulas of degree-based topological indices for families of graphs $C_4(K_n)$. The results are analyzed, and the general formulas are derived for these families of graphs.

Theorem 1. *The Randic and sum connectivity indices of families of graphs $C_4(K_n)$ are*

$$R(G) = \frac{n-1}{\sqrt{n^2-1}} + \frac{1}{n-1}\left(\frac{n^2-3n+2}{2}\right) + \frac{2}{\sqrt{2n+2}} + 1,$$
$$SCI(G) = \frac{n-1}{\sqrt{2n}} + \frac{2}{\sqrt{n+3}} + \frac{1}{\sqrt{2n-2}}\left(\frac{n^2-3n+2}{2}\right) + 1. \tag{11}$$

Proof. To find the Randic or connectivity index of $C_4(K_n)$ as shown in Figure 1, we first select a vertex u_1 on $C_4(K_n)$ of degree $n+1$. There are $(n-1)$ vertices $u_2, u_3, \cdots, u_n$ of degree $n-1$, which are adjacent to u_1. For the vertices u_1 and u_i, where $i = 2, 3, 4, \cdots, n$, the sum is obtained as

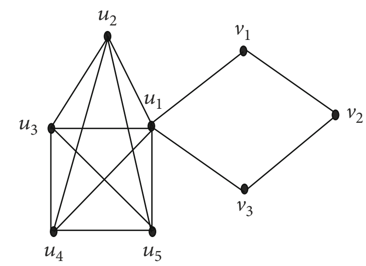

Figure 1: $G = C_4(K_5)$ (graph obtained by identifying one of the vertices of K_5 with one vertex of C_4).

$$\sum_{u_1\sim u_i} \frac{1}{\sqrt{d_{u_1}(G)d_{u_i}(G)}} = \frac{n-1}{\sqrt{(n+1)(n-1)}} i = 2, 3, ,\cdots, n = \frac{n-1}{\sqrt{n^2-1}}. \tag{12}$$

Since the degree of u_1 is $n+1$, the other two vertices which are adjacent to u_1, are v_1 and v_3 of degree two. For the vertices u_1 and v_1, we have

$$\frac{1}{\sqrt{d_{u_1(G)}d_{v_1(G)}}} = \frac{1}{\sqrt{2n+2}}. \tag{13}$$

Similarly, for the vertices u_1 are v_3, we have

$$\frac{1}{\sqrt{d_{u_1(G)}d_{v_3(G)}}} = \frac{1}{\sqrt{2n+2}}. \tag{14}$$

Now, we select a vertex u_2 in $C_4(K_n)$. Since the degree of u_2 is $n-1$, the other $(n-2)$ vertices which are adjacent to u_2 , are $u_3, u_4, \cdots, u_n$. For the vertices u_2 and u_j where $j = 3, 4, 5, \cdots, n$, the sum is obtained as

$$\sum_{u_2\sim u_j} \frac{1}{\sqrt{d_{u_2}(G)d_{u_j}(G)}} = \frac{n-2}{n-1}. \tag{15}$$

Since K_n is symmetric, the same result is obtained for the remaining $(n-2)$ vertices $u_3, u_4, \cdots, u_n$. So combining the result for all $(n-1)$ vertices $u_2, u_3, \cdots, u_n$, we have

$$= \frac{n-2}{n-1} + \frac{n-3}{n-1} + \frac{n-4}{n-1} + \cdots + \frac{n-(n-1)}{n-1} = \frac{1}{n-1}\left(\frac{n^2-3n+2}{2}\right). \tag{16}$$

Now, we select v_1 on $C_4(K_n)$. The degree of v_1 is two, the other vertex which is adjacent to v_1 is v_2 of degree two. For the vertices v_1 and v_2, we have

$$\frac{1}{\sqrt{d_{v_1(G)}d_{v_2(G)}}} = \frac{1}{2}. \quad (17)$$

Similarly, we select v_2 on $C_4(K_n)$. The degree of v_2 is two, the other vertex which is adjacent to v_2 is v_3 of degree two. For vertices v_2 and v_3, we have

$$\frac{1}{\sqrt{d_{v_2(G)}d_{v_3(G)}}} = \frac{1}{2}. \quad (18)$$

Adding equations (12) to (18), we have $R(G)$ = Randic or connectivity index of $C_4(K_n)$ is

$$R(G) = \frac{n-1}{\sqrt{n^2-1}} + \frac{1}{n-1}\left(\frac{n^2-3n+2}{2}\right) + \frac{2}{\sqrt{2n+2}} + 1. \quad (19)$$

Similarly, $\mathrm{SCI}(G)$ = sum connectivity index of $C_4(K_n)$ is

$$\mathrm{SCI}(G) = \frac{n-1}{\sqrt{2n}} + \frac{2}{\sqrt{n+3}} + \frac{1}{\sqrt{2n-2}}\left(\frac{n^2-3n+2}{2}\right) + 1. \quad (20)$$

□

Theorem 2. *The atom bond connectivity index and augmented Zagreb index of families of graphs $C_4(K_n)$ are defined, respectively, by*

$$\begin{aligned} ABC(G) &= (n-1)\sqrt{\frac{2}{n+1}} + \frac{\sqrt{2n-4}}{n-1}\left(\frac{n^2-3n+2}{2}\right) + 2\sqrt{2}, \\ AZI(G) &= (n-1)\left(\frac{n+1}{2}\right)^3 + \frac{(n-1)^6}{(2n-4)^3}\left(\frac{n^2-3n+2}{2}\right) + 32. \end{aligned} \quad (21)$$

The proof is the same as that of Theorem 1.

Theorem 3. *The geometric arithmetic index and harmonic index of families of graphs $C_4(K_n)$ are defined, respectively, by*

$$\begin{aligned} GA(G) &= \frac{(n-1)\sqrt{n^2-1}}{n} + \frac{4\sqrt{2n+2}}{n+3} + \frac{n^2-3n+2}{2} + 2, \\ H(G) &= \frac{n-1}{n} + \frac{4}{n+3} + \frac{1}{n-1}\left(\frac{n^2-3n+2}{2}\right) + 1. \end{aligned} \quad (22)$$

The proof is similar to the proof of Theorem 1.

Theorem 4. *The Zagreb indices of families of graphs $C_4(K_n)$ are*

$$\begin{aligned} M_1(G) &= n^3 - 2n^2 + 5n + 12, \\ M_2(G) &= n^3 - n^2 + 3n + 13 + \left[\frac{(n^2-3n+2)(n-1)^2}{2}\right]. \end{aligned} \quad (23)$$

Proof. For the family of graphs $C_4(K_n)$, there are $(n-1)$ vertices $u_2, u_3, \cdots, u_n$ of degree $n-1$, one vertex u_1 of degree $n+1$, and three vertices v_1, v_2, and v_3 of degree two, respectively. For the vertex u_1, we have

$$\deg(u_1)^2 = (n+1)^2. \quad (24)$$

For the vertices v_1, v_2 and v_3, we have

$$\deg(v_1)^2 + \deg(v_2)^2 + \deg(v_3)^2 = 12. \quad (25)$$

For vertices $u_2, u_3, \cdots, u_n$, we have

$$\deg(u_2)^2 + \deg(u_3)^2 + \cdots + \deg(u_n)^2 = (n-1)^3. \quad (26)$$

Adding equations (24) to (26), we have $M_1(G)$ = first Zagreb index of $C_4(K_n)$ is

$$M_1(G) = n^3 - 2n^2 + 5n + 12. \quad (27)$$

Similarly, $M_2(G)$ = second Zagreb index of $C_4(K_n)$ is

$$M_2(G) = n^3 - n^2 + 3n + 13 + \left[\frac{(n^2-3n+2)(n-1)^2}{2}\right]. \quad (28)$$

□

Theorem 5. *The Katayma and multiplicative Zagreb indices of families of graphs $C_4(K_n)$ are*

$$\begin{aligned} \prod_1(G) &= 64(n+1)^2\left(n^2-2n+1\right)^{n-1}, \\ \prod_2(G) &= 16\left(n^2-1\right)^{(n-1)}(2n+2)^2(n-1)^{\left(n^2-3n+2\right)}, \\ \prod_1^*(G) &= 16(2n)^{(n-1)}(3+n)^2(2n-2)^{\left(\left(n^2-3n+2\right)/2\right)}. \end{aligned} \quad (29)$$

Proof. Since there are $(n-1)$ vertices $u_2, u_3, \cdots, u_n$ of degree $n-1$, we have one vertex of degree $n+1$ and three vertices of degree two in $C_4(K_n)$ graph. Thus, we have

$$\begin{aligned} &\deg(u_2)^2 \deg(u_3)^2, \cdots, \deg(u_n)^2 \\ &\quad = (n-1)^2(n-1)^2, \cdots, (n-1)^2(n-1).times \\ &\quad = \left(n^2-2n+1\right)^{(n-1)}. \end{aligned} \quad (30)$$

Table 1: Topological indices of $G = C_4(K_6)$.

$\prod_1(G)$	3.0625×10^{10}
$H(G)$	4.278
$R(G)$	4.380
$\mathrm{SCI}(G)$	6.272
$\mathrm{ABC}(G)$	11.158
$\mathrm{GA}(G)$	18.593
$M_1(G)$	186

For the vertex u_1, we have

$$\deg(u_1)^2 = (n+1)^2. \quad (31)$$

Similarly, for the vertices v_1, v_2, and v_3, we have

$$\deg(v_1)^2 \deg(v_2)^2 \deg(v_3)^2 = (2)^2.(2)^2.(2)^2 = 64. \quad (32)$$

Multiplying equations (30) to (32), $\prod_1(G)$ =first multiplicative Zagreb index of $C_4(K_n)$ is

$$\prod_1(G) = 64(n+1)^2\left(n^2-2n+1\right)^{n-1}. \quad (33)$$

Similarly, second multiplicative and modified first multiplicative Zagreb indices of $C_4(K_n)$ are defined, respectively, by

$$\begin{aligned} \prod_2(G) &= 16\left(n^2-1\right)^{(n-1)}(2n+2)^2(n-1)^{\left(n^2-3n+2\right)}, \\ \prod_1^*(G) &= 16(2n)^{(n-1)}(3+n)^2(2n-2)^{\left(\left(n^2-3n+2\right)/2\right)}. \end{aligned} \quad (34)$$

□

Example 6. Topological indices of graph $C_4(K_6)$ are shown in Table 1.

3. Topological Indices of Families of Graphs $C_4(K_n) + v_1v_3$

By looking at the earlier results for computing the topological indices for different families of graphs, here, we introduce new degree-based topological indices to compute their values for families of graphs $C_4(K_n) + v_1v_3$.

Theorem 7. *The Randic and sum connectivity indices of families of graphs $C_4(K_n) + v_1v_3$ are defined, respectively, by*

$$\begin{aligned} R(G) &= \frac{n-1}{\sqrt{n^2-1}} + \frac{1}{n-1}\left(\frac{n^2-3n+2}{2}\right) + \frac{2}{\sqrt{3n+3}} + \frac{2}{\sqrt{6}} + \frac{1}{3}, \\ SCI(G) &= \frac{n-1}{\sqrt{2n}} + \frac{2}{\sqrt{n+4}} + \frac{1}{\sqrt{2n-2}}\left(\frac{n^2-3n+2}{2}\right) + \frac{2}{\sqrt{5}} + \frac{1}{\sqrt{6}}. \end{aligned} \quad (35)$$

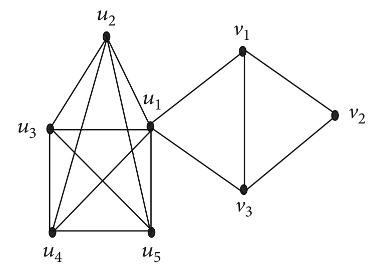

Figure 2: $G = C_4(K_5) + v_1v_3$ (graph obtained by identifying one of the vertices of K_5 with one vertex of $C_4 + v_1v_3$).

Proof. To find the Randic or connectivity index of $C_4(K_n) + v_1v_3$ as presented in Figure 2, we first select a vertex u_1 on $C_4(K_n) + v_1v_3$ of degree $n+1$. There are $(n-1)$ vertices $u_2, u_3, \cdots, u_n$ of degree $n-1$, which are adjacent to u_1. For the vertices u_1 and u_i where $i = 2, 3, 4, \cdots, n$, the sum

$$\sum_{u_1 \sim u_i} \frac{1}{\sqrt{d_{u_1}(G) d_{u_i}(G)}}, i = 2, 3, 4, \cdots, n, \quad (36)$$

is obtained as

$$= \frac{n-1}{\sqrt{(n+1)(n-1)}} = \frac{n-1}{\sqrt{n^2-1}}. \quad (37)$$

Since the degree of u_1 is $n+1$, the other two vertices which are adjacent to u_1, are v_1 and v_3 of degree 3. For the vertices u_1 and v_1, we have

$$\frac{1}{\sqrt{d_{u_1(G)} d_{v_1(G)}}} = \frac{1}{\sqrt{3n+3}}. \quad (38)$$

Similarly, for the vertices u_1 are v_3, we have

$$\frac{1}{\sqrt{d_{u_1(G)} d_{v_3(G)}}} = \frac{1}{\sqrt{3n+3}}. \quad (39)$$

Now, we select a vertex u_2 on $C_4(K_n)$. Since the degree of u_2 is $n-1$, the other $(n-2)$ vertices which are adjacent to u_2, are $u_3, u_4, cdots, u_n$. For the vertices u_2 and u_j where $j = 3, 4, 5, \cdots, n$, the sum is obtained as

$$\sum_{u_2 \sim u_j} \frac{1}{\sqrt{d_{u_2}(G) d_{u_j}(G)}} = \frac{n-2}{n-1}. \quad (40)$$

Since K_n is symmetric, the same result is obtained for the remaining $n-2$ vertices $u_3, u_4, \cdots, u_n$. By using equation (16), the above equation can be written as

$$= \frac{1}{n-1}\left(\frac{n^2-3n+2}{2}\right). \quad (41)$$

Now, we select v_1 on $C_4(K_n)$. Since the degree of v_1 is three, the other two vertices which are adjacent to v_1, are v_2 and v_3. For vertices v_1 and v_2, we have

$$\frac{1}{\sqrt{d_{v_1(G)}d_{v_2(G)}}} = \frac{1}{\sqrt{6}}. \tag{42}$$

Similarly, for the vertices v_1 and v_3, we have

$$\frac{1}{\sqrt{d_{v_1(G)}d_{v_3(G)}}} = \frac{1}{3}. \tag{43}$$

Now, we select v_2 on $C_4(K_n)$. The degree of v_2 is two, the other vertex which is adjacent to v_2, is v_3 of degree three. For vertices v_2 and v_3, we have

$$\frac{1}{\sqrt{d_{v_2(G)}d_{v_3(G)}}} = \frac{1}{\sqrt{6}}. \tag{44}$$

Adding equations (37) to (44) $R(G)$ = Randic or connectivity index of $C_4(K_n)+v_1v_3$ is $R(G)=(n-1)/\sqrt{n^2-1}+(1/(n-1))((n^2-3n+2)/2)+(2/\sqrt{2n+2})+1$.

Similarly, $SCI(G)$ = sum connectivity index of $C_4(K_n)+v_1v_3$ is

$$\mathrm{SCI}(G) = \frac{n-1}{\sqrt{2n}} + \frac{2}{\sqrt{n+4}} + \frac{1}{\sqrt{2n-2}}\left(\frac{n^2-3n+2}{2}\right) + \frac{2}{\sqrt{5}} + \frac{1}{\sqrt{6}}. \tag{45}$$

Using the same arguments as in Theorem 6, we determine some other topological indices of $C_4(K_n)+v_1v_3$. □

Theorem 8. *The atom bond connectivity index and augmented Zagreb index of families of graphs $C_4(K_n)+v_1v_3$ are defined, respectively, by*

$$\begin{aligned} ABC(G) &= (n-1)\sqrt{\frac{2}{n+1}} + \frac{\sqrt{2n-4}}{n-1}\left[\frac{n^2-3n+2}{2}\right] \\ &\quad + 2\sqrt{\frac{n+2}{3n+3}} + \sqrt{2} + \frac{2}{3}, \\ AZI(G) &= (n-1)\left(\frac{n+1}{2}\right)^3 + \frac{(n-1)^6}{(2n-4)^3}\left(\frac{n^2-3n+2}{2}\right) \\ &\quad + 2\left(\frac{3n+3}{n+2}\right)^3 + \frac{1753}{64}. \end{aligned} \tag{46}$$

Theorem 9. *The geometric arithmetic index and harmonic index of families of graphs $C_4(K_n)+v_1v_3$ are defined, respectively, by*

$$\begin{aligned} GA(G) &= \frac{(n-1)\sqrt{n^2-1}}{n} + \frac{4\sqrt{3n+3}}{n+4} + \frac{n^2-3n+2}{2} + \frac{4\sqrt{6}}{5} + 1 \\ H(G) &= \frac{n-1}{n} + \frac{4}{n+4} + \frac{1}{n-1}\left(\frac{n^2-3n+2}{2}\right) + \frac{17}{15}. \end{aligned} \tag{47}$$

Theorem 10. *The Zagreb indices of the families of graphs $C_4(K_n)+v_1v_3$ are*

$$\begin{aligned} M_1(G) &= n^3 - 2n^2 + 5n + 22, \\ M_2(G) &= n^3 - n^2 + 5n + 28 + \left[\frac{(n^2-3n+2)(n-1)^2}{2}\right]. \end{aligned} \tag{48}$$

Proof. For the families of graphs $C_4(K_n)+v_1v_3$ there are $n-1$ vertices $u_2, u_3 \ldots \cdots u_n$ of degree $n-1$, one vertex u_1 of degree $n+1$ and two vertices v_1 and v_3 of degree three and one vertex v_2 of degree two, respectively.

$$\deg(u_1)^2 = (n+1)^2. \tag{49}$$

For vertices v_1 and v_3, the first Zagreb index is

$$\deg(v_1)^2 + \deg(v_3)^2 = (3)^2 + (3)^2 = 18. \tag{50}$$

For vertices $u_2, u_3 \ldots \cdots u_n$, we have

$$\begin{aligned} &\deg(u_2)^2 + \deg(u_3)^2 + \cdots \cdots \cdots + \deg(u_n)^2 \\ &\quad = (n-1)^2 + (n-1)^2 + . \cdots \cdots (n-1)^2 (n-1)\text{times} \\ &\quad = (n-1)(n-1)^2 = (n-1)^3. \end{aligned} \tag{51}$$

Similarly, for vertex v_2 we have

$$\deg(v_2)^2 = 2^2 = 4. \tag{52}$$

Adding equations (49) to (52), we have $M_1(G)$ = first Zagreb index of $C_4(K_n)+v_1v_3$

$$M_1(G) = (n+1)^2 + (n-1)^3 + 22 = n^3 - 2n^2 + 5n + 22. \tag{53}$$

Similarly, $M_2(G)$ = second Zagreb index of $C_4(K_n)+v_1v_3$

$$M_2(G) = n^3 - n^2 + 5n + 28 + \left[\frac{(n^2-3n+2)(n-1)^2}{2}\right]. \tag{54}$$

□

Table 2: Topological indices of $G = C_4(K_6) + v_1v_3$.

$\prod_1(G)$	1.5503×10^{11}
$H(G)$	4.366
$R(G)$	4.431
$SCI(G)$	6.541
$ABC(G)$	10.231
$GA(G)$	19.723
$M_1(G)$	196

Theorem 11. *The Katayma and multiplicative Zagreb indices of families of graphs $C_4(K_n) + v_1v_3$ are*

$$\begin{aligned} &\prod_1(G) = 324(n+1)^2\left(n^2-2n+1\right)^{n-1}, \\ &\prod_2(G) = 324\left(n^2-1\right)^{(n-1)}(3n+3)^2(n-1)^{\left(n^2-3n+2\right)}, \\ &\prod_1^*(G) = 150(2n)^{(n-1)}(n+4)^2(2n-2)^{\left(\left(n^2-3n+2\right)/2\right)}. \end{aligned} \tag{55}$$

Proof. Since there are $n-1$ vertices $u_2, u_3 \cdots u_n$ of degree $n-1$, one vertex of degree $n+1$ and three vertices of degree two in $C_4(K_n) + v_1v_3$ graph. Thus, we have

$$\begin{aligned} &\deg(u_2)^2 \deg(u_3)^2 \cdots \cdots \deg(u_n)^2 \\ &\quad = (n-1)^2(n-1)^2 \cdots \cdots (n-1)^2(n-1)\text{times} \\ &\quad = \left(n^2-2n+1\right)^{(n-1)}. \end{aligned} \tag{56}$$

For vertex u_1, we have

$$\deg(u_1)^2 = (n+1)^2. \tag{57}$$

Similarly, for vertices v_1 and v_3, we have

$$\deg(v_1)^2 \deg(v_3)^2 = (3)^2.(3)^2 = 81. \tag{58}$$

Also, for vertex v_2, we have

$$\deg(v_2)^2 = (2)^2 = 4. \tag{59}$$

Multiplying equations (56) to (59), we have $\prod_1(G) =$ first multiplicative Zagreb index of $C_4(K_n) + v_1v_3$

$$\prod_1(G) = 324(n+1)^2\left(n^2-2n+1\right)^{n-1}. \tag{60}$$

Similarly, second multiplicative Zagreb index and modified first multiplicative Zagreb index of $C_4(K_n) + v_1v_3$ are

$$\begin{aligned} &\prod_2(G) = 324\left(n^2-1\right)^{(n-1)}(3n+3)^2(n-1)^{\left(n^2-3n+2\right)} \\ &\prod_1^*(G) = 150(2n)^{(n-1)}(n+4)^2(2n-2)^{\left(\left(n^2-3n+2\right)/2\right)}. \end{aligned} \tag{61}$$

□

Example 12. Topological indices of graph $C_4(K_6) + v_1v_3$ are shown in Table 2.

4. Concluding Remarks

This research work was done to understand the relationship between various concepts about topological indices and graphs. In this article, we have studied certain vertex degree-based topological indices and derived the closed formulas of these indices for two special families of graphs, i.e., $C_4(K_n)$ and $C_4(K_n) + v_1v_3$. It was easily checked that all vertex degree-based topological indices of families of these two graphs remain the same for all values of n and the topology of a graph is completely changed if we add an additional edge in it. In the future, we are interested in investigating and calculating some other topological indices of two more special families of graphs.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Disclosure

The statement made and views expressed are solely the responsibility of the author.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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References

[1] A. M. Hanan Ahmed, A. Alwardi, and M. R. Salestina, "On domination topological indices of graphs, International Journal of," *Analysis and Applications*, vol. 19, pp. 47–64, 2021.

[2] F. Deng, X. Zhang, M. Alaeiyan, A. Mehboob, and M. R. Farahani, "Topological indices of the pent-heptagonal nanosheets

VC5C7 and HC5C7," *Advances in Materials Science and Engineering*, vol. 2019, Article ID 9594549, 12 pages, 2019.

[3] M. S. Abdelgader, C. Wang, and S. A. Mohammed, "Computation of topological indices of some special graphs," *Mathematics*, vol. 6, p. 33, 2018.

[4] S. Mufti, M. F. Nadeem, W. Gao, and Z. Ahmad, "Topological study of the para-line graphs of certain pentacene via topological indices," *Open Chemistry*, vol. 16, no. 1, pp. 1200–1206, 2018.

[5] I. Gutman, "Degree- based topological indices," *Croatica Chemica Acta*, vol. 86, no. 4, pp. 351–361, 2013.

[6] S. Mondal, D. Nilanjan, and P. Anita, "Topological indices of some chemical structures applied for the treatment of COVID-19 patients," *Polycyclic Aromatic Compounds*, pp. 1–15, 2020.

[7] A. Ayache and A. Alame, "Topological indices of the -graph," *Journal of the Association of Arab Universities for Basic and Applied Sciences*, vol. 24, no. 1, pp. 283–291, 2017.

[8] X. Zuo, J. Liu, I. Hifza, K. Ali, and S. T. Raza Rizvi, "Topological indices of certain transformed chemical structures," *Journal of Chemistry*, vol. 2020, Article ID 3045646, 7 pages, 2020.

[9] X. Li and Y. Shi, "A survey on the Randic index," *MATCH Communications in Mathematical and in Computer Chemistry*, vol. 59, no. 1, pp. 127–156, 2008.

[10] I. G. Farkoush, M. Alaeiyan, M. R. Farahani, and M. Maghasedi, "Computing the Narumi–Katayama index and modified Narumi–Katayama index of some families of dendrimers and tetrathiafulvalene," *Journal of Mathematics*, vol. 2021, Article ID 8492991, 3 pages, 2021.

[11] D. S. Nandappa, M. R. R. Kanna, and P. R. Kumar, "Narumi-Katayama and multiplicative Zagreb indices of Dutch windmill graph," *International Journal of Scientific and Engineering Research*, vol. 7, pp. 32-33, 2016.

[12] E. D. Molina, J. M. Rodriguez, J. L. Sanchez, and J. M. Sigarreta, "Some properties of the arithmetic-geometric index," *Symmetry*, vol. 13, no. 5, p. 857, 2021.

[13] K. Das, I. Gutman, and B. Furtulab, "On atom-bond connectivity index," *Filomat*, vol. 26, pp. 733–738, 2012.

[14] B. Furtula, A. Graovac, and D. Vukičević, "Augmented Zagreb index," *Journal of Mathematical Chemistry*, vol. 48, no. 2, pp. 370–380, 2010.

[15] A. Ali, Z. Raza, and A. A. Bhatti, "On the augmented Zagreb index," 2014, https://arxiv.org/abs/1402.3078.

[16] L. Zhong, "The harmonic index for graphs," *Applied Mathematics Letters*, vol. 25, no. 3, pp. 561–566, 2012.

[17] B. S. Shetty, V. Lokesha, and P. S. Ranjini, "On the harmonic index of graph operations," *Transactions on combinatorics*, vol. 4, pp. 5–14, 2015.

[18] Z. Du and B. Zhou, "On sum-connectivity index of bicyclic graphs," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 35, pp. 101–117, 2012.

[19] H. S. Ramanea, V. V. Manjalapura, and I. Gutman, "General sum-connectivity index, general product-connectivity index, general Zagreb index and coindices of line graph of subdivision graphs," *AKCE International Journal of Graphs and Combinatorics*, vol. 14, no. 1, pp. 92–100, 2017.

[20] A. Vasilyev, *Upper and lower bounds of symmetric division deg index*, 2014.

[21] A. Ali, S. Elumalai, and T. Mansour, "On the symmetric division deg index of molecular graphs," *MATCH Communications in Mathematical and in Computer Chemistry*, vol. 83, pp. 205–220, 2020.