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Functioning of Divine Ratio on Conformal Einstein Spaces

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Abstract

Albert Einstein, during the development of his precious theory of relativity evoked a fascinating proverb; *"Two things are infinite in this world! One is the beauty of universe and the other is human stupidity. Though! I am not sure about the beauty of universe."* Perhaps, the inspirational notion from the aforementioned proverb may yield in the form of DIVINE RATIO, the magical proportion. When one talks about the beauty of universe from the stand point of Mathematics, one would probably think about this enthusiastic number. The present article is intended to discuss the compatibility of Divin structure in Einstein as well as conformal Einstein spaces. A brief research on the compatibility of Divine structure with many well known Einstein as well as conformal Einstein equations has been carried out and based on the compatibility conditions, new kinds of spaces e.g., Divine Einstein and conformally Divine Einstein spaces have been generated. Moreover, some new tensor fields e.g., Divine Yang, Divine Bach etc. and the three new looking conformally Divine Einstein equations have also been established.

Keywords: Divine; Bach; Weyl; Yang; Ricci; eigenvalue; Divine-Einstein
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1 Introduction

The great scientist Johannes Kepler (1571-1630) had evoked a fascinating stanza: *Geometry has two great treasures; one is the theorem of Pythagoras and the other is the division of line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel.*" The Golden ratio, synonymously cited as Golden proportion, Golden number, Golden mean, Golden section, Divine ratio, Divine section, Golden cut, mean of the Phidias and Divine proportion,

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has been a well known irrational number $1.6180339887\cdots$, in this planet, which has an enthusiastic mystic behavior to almost living and non-living things of the world. The Golden ratio was evolved by the famous founder of Geometry, i,e., Euclid as a formalized system around 300 BC. At the time the Golden ratio was discovered in Greece by Euclid, the Greece was experiencing a golden era and the citizens of Greece were typically known for their elegant art and architecture. Really, Euclid exposed a visually pleasing geometric ratio, which has been confirmed as the composition of beauty by many artists, architects and mathematicians. However, from the ancient literature concerning the Greece, it has been found that the discovery of Divine ratio is attributed to Pythagoras or his followers [Fischler (1998)]. But at the ground root, around 300 BC, Euclid had provided the very first written literature on the Divine ratio as a line divided into extreme and mean ratio. Since the discovery of Divine proportion till the date, a huge amount of research has been carried out by researchers and it has been found that the Divine proportion appears almost everywhere in this universe. For instance, the Golden ratio appears in the dimension of human body, structure of musical compositions and in the ratio of harmonious sound frequencies [Livo (2002)].

Besides, many natural and human manufactured objects in this world have some mathematical descriptions due to Divine ratio, for example, the petal arrangement of roses and the objects having pentagonal symmetry such as, inflorescence of many flowers and phyllotaxis objects have some numerical exposition due to Fibonacci numbers which are themselves defined as the Divine proportions [Crâşmăreanu and Hreţcanu (2008)]. Moreover, out of the Divine proportion, there arises a Divine spiral whose familiar coil shape can be seen everywhere in nature, such as in the DNA structure and fingerprints, sun flowers and seashells, storm clouds and tornados. Nevertheless, the prime examples of the use of Golden ratio in architecture are the great pyramid of Giza (built around 2560 BC)[Fischler (2000)] and the holy temple of Parthenos in Greece, built and decorated between 432-447 BC. Let us now recall a brief digest regarding Divine ratio.

In Mathematics and nature, two quantities are said to be in Divine ratio, if the ratio of the sum of quantities to larger quantity is equal to the ratio of the larger quantity to the smaller one. Euclid's definition concerning to Golden ratio states that "A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the lesser" [Encyclopedia (2011)], [Fischler (2000)]. Moreover, according to Crâşmăreanu and Hreţcanu (2008), the Golden proportion partitions a line segment into a major subsegment and a minor subsegment in such a fashion that the proportion of entire line segment and major subsegment and the proportion of major subsegment and minor subsegment must be equal to the number ϕ (the Phidias number). The first Greek letter ϕ (read as 'phi') is used for Divine proportion in the honor of great Greek sculptor Phidias who lived around 450 BC [Fischler (1998)]. The Golden ratio, which is indicated by ϕ is the real positive root of the quadratic equation $x^2 - x - 1 = 0$ and has the value $\phi = \frac{1+\sqrt{5}}{2} = 1.6180339887\cdots$. Furthermore, the quadratic equation $x^2 - x - 1 = 0$ is called the "Fibonacci" or "Golden" equation. Euclid has defined the Golden ratio as a line cut in extreme and mean ratio and the proportion is now often seen in simple geometric figures like pentagram, decagon and dodecagon etc.

The equation $x^2 - x - 1 = 0$, which is usually called Fibonacci has two real solutions ϕ and its conjugate $\bar{\phi} = 1 - \phi$. It is noteworthy that the Divine ratio and its conjugate are connected in the following fashion [Seiichi and Akifumi (2008)]:

$$\begin{cases} \phi + \bar{\phi} = 1 \\ \phi \cdot \bar{\phi} = -1 \\ \phi^{-1} = \phi - 1 \;, \\ (\bar{\phi})^{-1} = -\phi \\ \phi^{-1} + (\bar{\phi})^{-1} = -1 \end{cases} \tag{1.1}$$

and

$$\begin{cases} \phi^{-1} \cdot (\bar{\phi})^{-1} = -1 \\ \phi^2 = 1 + \phi \\ \bar{\phi}^2 = 2 - \phi \; . \\ \phi^2 + \bar{\phi}^2 = 3 \\ \phi^2 \cdot \bar{\phi}^2 = 1 \end{cases} \tag{1.2}$$

The intention of the present article is to investigate the compatibility and functioning of Divine ratio and hence the Divine structure for conformal Einstein's spaces. For this purpose and to obtain new expressions for Einstein's theory of relativity, we have concerned the precious articles [Hreţcanu and Crâşmăreanu (2009)], [Crâşmăreanu and Hreţcanu (2008)]. Even the considerable amount of research pertaining Golden differential geometry and applications of Golden ratio on Riemannian manifolds has been carried out by Hreţcanu and Crâşmăreanu (2009) and Crâşmăreanu and Hreţcanu (2008), then also to meet our purpose, in the following subsection, we outline some prerequisites discussed by Hreţcanu and Crâşmăreanu (2009) and Crâşmăreanu and Hreţcanu (2008).

1.1 Structures on Manifolds Due to Golden Ratio: Divine Structures

The notions of constructing a Golden structure on a C^∞-differentiable real manifold as well as on a Riemannian manifold have been extensively discussed by Hreţcanu and Crâşmăreanu (2009), Crâşmăreanu and Hreţcanu (2008) and Hreţcanu and Crâşmăreanu (2007). In order to evolve such a beautiful structure, the concepts of $f-structures$ [Yano and Kon (1984)] and their extended form, i.e., *polynomial structures* [Goldberg and Yano (1970)] on a manifold have been consulted. It is mentioned that a polynomial structure on a manifold looks like a C^∞ tensor field F of type $(1,1)$, which has been defined on a differentiable manifold N in such a way that the following algebraic equation is satisfied [Hreţcanu and Crâşmăreanu (2009)]:

$$Q(x) = x^n + a_n x^{n-1} + \cdots + a_2 x + a_1 I = 0, \tag{1.3}$$

where I stands for identity mapping and for $x = F$, $F^{n-1}(p), F^{n-2}(p), \cdots, F(p), I$ are linearly independent for every $p \in N$.
From the above detail, Crâşmăreanu and Hreţcanu (2008) has developed a beautiful concise definitions as follows:

Definition 1.1 (Goldberg and Yano (1970))**.** Let N be a C^∞-differentiable real manifold. A tensor field F of the type (1,1) on M is said to define a polynomial structure if F satisfies the algebraic equation:

$$Q(x) = x^n + a"_n x^{n-1} + \cdots + a_2 x + a_1 I = 0,$$

where $F^{n-1}(p), F^{n-2}(p), \cdots, F(p), I$ are linearly independent $\forall\; p \in N$.

Also, for $Q(x) = x^2 + 1$ (or $Q(x) = x^2 - 1$), one can obtain an almost complex structure J (respectively, an almost product structure P). Further, Crâşmăreanu and Hreţcanu (2008) delineated that existence of an almost complex structure induces a condition on the dimension of N, such as it is even. For $Q(x) = x^2$, the concept of almost tangent structure T [Miron and Anastasiei (1994)] has also been suggested by Crâşmăreanu and Hreţcanu (2008).
On the basis of [Goldberg and Petridis (1973)], an almost product structure P for N has been generated by Hreţcanu and Crâşmăreanu (2009). For more details regarding the establishment of such complex structures, readers are suggested to refer [Hreţcanu and Crâşmăreanu (2009)], [Crâşmăreanu and Hreţcanu (2008)] and [Hreţcanu and Crâşmăreanu (2007)].

From what has been follows, Hreţcanu and Crâşmăreanu (2009) and Crâşmăreanu and Hreţcanu (2008) have mentioned a Divine structure as a polynomial structure along with the structure polynomial of the type $Q(x) = x^2 - x - I$ as below:

Definition 1.2 (Hreţcanu (2007))**.** A Golden structure on the manifold N of dimension n is a tensor field G of type $(1,1)$, which satisfies the equation:

$$G^2 = G + I. \tag{1.4}$$

The following features of Golden structure G have also been worked out by Hreţcanu and Crâşmăreanu (2009) and Crâşmăreanu and Hreţcanu (2008):

Proposition 1.1. *A Golden structure G on an n-dimensional manifold N has the power*

$$G^n = F_n G + F_{n-1} I, \tag{1.5}$$

where n is any integer and $(F_n)_n$ is the well known Fibonacci sequence.

By making use of explicit expression for the Fibonacci (i.e., the Binet's formula) [Livo (2002)];

$$F_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}, \tag{1.6}$$

the new version of (1.5) has been evolved as

$$G^n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}} G + \frac{\phi^{n-1} - (1-\phi)^{n-1}}{\sqrt{5}} I. \tag{1.7}$$

Proposition 1.2 (Hreţcanu and Crâşmăreanu (2009), Crâşmăreanu and Hreţcanu (2008))**.** ....

1. *The eigenvalues of a Golden structure G are the Golden ratio ϕ and $(1-\phi)$.*
2. *The Golden structure G is the isomorphism on the tangent space T_xN of manifold N, $\forall\, x \in N$.*
3. *It follows that G is invertible and its inverse $\hat{G} \equiv G^{-1}$ satisfies:*

$$\hat{G}^2 = -\hat{G} + I. \tag{1.8}$$

It has been notified by Hreţcanu and Crâşmăreanu (2009) and Crâşmăreanu and Hreţcanu (2008) that Golden structure appears in pairs; namely if G is a Golden structure then $\bar{G} = I - G$ is also a Golden structure. But that in case of almost tangent structures (T and $-T$), almost complex structures (J and $-J$) and for the almost product structures (P and $-P$), it is obvious to seek for a connection between Golden structure and product structures.

Theorem 1.1 (Crâşmăreanu and Hreţcanu (2008))**.** *An almost product structure P induces a Golden structure as below:*

$$G = \frac{1}{2}(1 + \sqrt{5}P). \tag{1.9}$$

Conversely, any Golden structure G yields an almost product structure,

$$P = \frac{1}{\sqrt{5}}(2G - I). \tag{1.10}$$

In the above mentioned relation it is evident that $\bar{G} = I - G \leftrightarrow \bar{P} = -P$. Likewise, if N be endowed with an almost tangent structure T, one can say that

$$G_t = \frac{1}{2}(I + \sqrt{5}T) \tag{1.11}$$

is a tangent Golden structure on the manifold (N, T). Also the equation

$$G_t^2 - G_t + \frac{1}{4}I = 0 \tag{1.12}$$

is the equation verified by tangent Golden structure.
The complex Golden structure G_c has also been delineated by Crâşmăreanu and Hreţcanu (2008) as follows:

Definition 1.3. Let (N, J) be an almost complex manifold. Then the tensor field G_c defined by;

$$G_c = \frac{1}{2}(I + \sqrt{5}J) \tag{1.13}$$

is called the complex Golden structure on N, J.

Moreover, the polynomial equation justified by G_c is given as;

$$G_c^2 - G_c + \frac{3}{2}I. \tag{1.14}$$

Hreţcanu and Crâşmăreanu (2009) and Crâşmăreanu and Hreţcanu (2008) have constructed the Golden structure on Riemannian manifolds and produced various delighting consequences. They have defined the Golden structure G on the Riemannian manifold as below:

Definition 1.4 (Hreţcanu and Crâşmăreanu (2009), Crâşmăreanu and Hreţcanu (2008))**.** The Golden Riemannian structure is a pair (g, G), which satisfies the compatibility condition:

$$g(GX, Y) = g(X, GY), \forall \text{ tangent vector fields } X, Y \in T_xN. \tag{1.15}$$

Also, a Riemannian manifold (N, g), endowed with a Golden structure G, such that the Riemannian metric g is G-compatible is known as Golden Riemannian manifold and symbolized as a triple (N, g, G).

2 Divine Structure on Conformal Einstein SPACES

Before discussing the functioning of Divine proportion as well as Divine structure on conformal Einstein spaces, let us go through the basic notions concerning Einstein and conformal Einstein spaces and outline some noteworthy expressions which are the necessary pre-requisites to pursue the proposed objectives.

It has been a wide spread fact that the Riemannian manifold (M, g) has feedback of three main curvatures. The first one is the Riemannian curvature tensor R^i_{jkl} itself, which is equivalent to sectional curvature function on some tangent plane. This curvature has the biquadratic form that provides adequate information from the curvature stand point. The second is Ricci curvature (or, contracted curvature) R_{ij}, which is obtained by taking trace of Riemannian curvature with respect to metric g_{ij}. Finally, the scalar curvature R, known as a scalar function on M, which can be obtained by taking trace of R_{ij} with respect to metric tensor g_{ij}.

It is also remarkable that whenever the dimension n of the manifold equals to 3, the Ricci tensor bears adequate information as the Riemannian tensor.

Now, a Riemannian or a pseudo Riemannian manifold (M, g) of dimension $n \geq 2$, is said to be an Einstein manifold (E, g), if the Ricci tensor R_{ij} is proportional to the metric tensor g_{ij}, i.e.,

$$R_{ij} = \mu g_{ij}, \tag{2.1}$$

where μ is a constant whose value can be calculated by transvecting (2.1) throughout with the inverse metric tensor g^{ij}.

Thus the equation (2.1) becomes

$$R_{ij} = \frac{R}{n} g_{ij}. \tag{2.2}$$

By normalization, we can assume the following three possibilities due to equation (2.1).

i $R_{ij} = g_{ij}$ (when μ is positive)

ii $R_{ij} = -g_{ij}$ (when μ is negative)

iii $R_{ij} = 0$ (when $\mu = 0$).

Here the corresponding number $\mu = 1, -1, 0$ is used to identify the sign of Einstein manifold.

In order to define the conformal Einstein spaces, let us rescale the Einstein metric g_{ij} (involved in equations (2.1) and (2.2)) under conformal transformation. Since the basic notion of conformal Einstein space comes out from the concept of transformation that preserves the angle and sense of the angle, thereby the Einstein metric $\widetilde{g}_{ij}$ is the conformal transformation of g_{ij}, if

$$\widetilde{g}_{ij} = e^{2\psi} g_{ij}, \tag{2.3}$$

where ψ is any arbitrary smooth scalar function called the "conformal parameter".
Moreover, by requiring that

$$\widetilde{g}^{ij}\widetilde{g}_{kj} = \delta^i_k = g^{ij} g_{kj}. \tag{2.4}$$

One can also obtain the rescaling of inverse Einstein metric as below:

$$\widetilde{g}^{ij} = e^{-2\psi} g^{ij}. \tag{2.5}$$

Remark 2.1. One can also formulate the conformal rescaling as;

$$\widetilde{g}_{ij} = \Omega^2 g_{ij}, \tag{2.6}$$

where Ω is called the conformal factor.

Comparison between (2.3) and (2.6) yields the relation between conformal factor and conformal parameter as $\psi = \ln \Omega$. Throughout the study, we shall use both of the above versions of conformal rescaling.

Remark 2.2. As far as we deal with space-time continuum of Einstein, the space-time coordinates x^i and the conformal factor Ω or equivalently ψ will in general depend on the other parameters. In particular, one can think if conformal rescaling of the form;

$$\widetilde{g}_{ij}(x^i) = \Big[\Omega(x^i, s, s^*)\Big]^2 g_{ij}[x^i, s, x^*], \tag{2.7}$$

where the parameters (s, s^*) are complex stereographic coordinates on the 2-sphere.

Now, with the foregoing conformal rescaling theory, we define the conformal Einstein space as follows:

Definition 2.1. An n-dimensional space with the metric g_{ij} is said to be conformal Einstein space $(\tilde{E}, \tilde{g})$, if $\exists$ a conformal rescaling of the form (2.3) or (2.6), such that

$$\widetilde{R}_{ij} = \frac{\widetilde{R}}{n}\widetilde{g}_{ij}. \tag{2.8}$$

Further, to the rescaled Einstein metric $\widetilde{g}_{ij}$, we define a metric compatible connection $\widetilde{\nabla}_i$ (which is assumed to be of Levi-Civita type and torsion free) such that

$$\widetilde{\nabla}_k \widetilde{g}_{ij} = 0 \tag{2.9}$$

and

$$\left(\widetilde{\nabla}_i\widetilde{\nabla}_j - \widetilde{\nabla}_j\widetilde{\nabla}_i\right) f = 0 \tag{2.10}$$

for all scalar field f.
We, now, define a polynomial structure to an n-dimensional Einstein manifold (E, g), which is called Divine structure (pre-defined by [Hreţcanu and Crâşmăreanu (2009), Crâşmăreanu and Hreţcanu (2008), Hreţcanu and Crâşmăreanu (2007)]) as a $(1, 1)$ tensor field like below:

Definition 2.2. A Divine structure G on an n-dimensional Einstein space (E, g) verifies the condition;

$$G^2 = G^i_j G^j_i = G^i_j + \delta^i_j, \tag{2.11}$$

where δ^i_j is the well known Kronecker tensor. The same condition in global coordinate system has been defined by Hreţcanu and Crâşmăreanu (2009) as,

$$G^2 = G + I, \tag{2.12}$$

where I is an identity operator on the Lie algebra $\chi(E)$ of vector fields on E.

Using (2.11) or (2.12), one can delineate the Divine Einstein structure (g, G) as follows:

Definition 2.3. The Divine Einstein structure is a pair (g, G) which satisfies the compatibility condition:

$$g_{ir} G^r_j = g_{rj} G^r_i, \tag{2.13}$$

where G^i_js are the components of Divine structure G. However, the same compatibility condition in global coordinate system has been revealed by Hreţcanu and Crâşmăreanu (2009) as

$$g(G(U), V) = g(U, G(V)), \tag{2.14}$$

where U and V are supposed to be tangent vector fields $\in \chi(E)$.

The compatibility condition (2.13) can also be expressed as

$$g_{rs} G^r_i G^s_j = g_{rj} G^r_i + g_{ij}, \tag{2.15}$$

or in global system [Hreţcanu and Crâşmăreanu (2009)],

$$g(G(U), G(V)) = g(G(U), V) + g(U, V), \tag{2.16}$$

for every tangent vector fields $U, V \in \chi(E)$.
It is also a remarkable aspect that the Einstein metric g is invariant under the action of Divine structure G), i.e.,

$$g_{rs} G^r_i G^s_j = g_{ij} \text{ or } g(G(U), G(V)) = g(U, V), \tag{2.17}$$

for every tangent vector fields $U, V \in \chi(E)$.
By considering the Divine Einstein structure (2.13), let us now define Divine Einstein manifold as follows:

Definition 2.4. An Einstein manifold (E, g) intimated with a Divine structure (g, G) is called a Divine Einstein manifold. Thus a tripe (E, g, G) in which an Einstein metric g and a Divine structure G compatible with g are involved, is called a Divine Einstein manifold.

With the aid of the propositions (2.1) & (1.1) of [Hreţcanu and Crâşmăreanu (2009)] and [Crâşmăreanu and Hreţcanu (2008)] respectively, we can prove the following:

Proposition 2.1. *A Divine Einstein manifold (E, g, G) bears the property*

$$G^n = F_n G^i_j + F_{n-1} \delta^i_j, \tag{2.18}$$

for any integer number $n > 0$. Here $(F_n)_n$ is the well known Fibonacci sequence.

Proof. From equation (2.11), it is easy to get

$$G^3 = 2G^i_j + \delta^i_j,$$

and in general, if we suppose that

$$G^{n+1} = F_n G^2 + F_{n-1} G^i_j = (F_n + F_{n-1}) G^i_j + F_n \delta^i_j,$$

which due to Fibonacci properties evidently produces (2.18). □

Proposition 2.2. *The Divine Einstein structure (g, G) defined for an n-dimensional Einstein manifold (E, g) bears the trace property, given as*

$$\text{trace}\,(G^2) = \text{trace}\,(G) + n. \tag{2.19}$$

.

Proof. The equation (2.19) can be evidently derived from (2.11), if we operate (2.11) by trace operator and note that trace $\delta^i_j = n$. However, if we use the concept of orthonormal basis $(E_1, E_2, \cdots E_m)$ of tangent space $T_x(E)$ at a point $x \in E$ [Hreţcanu and Crâşmăreanu (2009)], we have from (2.12)

$$g(G^2E_i, E_i) = g(GE_i, E_i) + g(E_i, E_i). \tag{2.20}$$

Summing (2.20) with respect to i, the equation (2.19) automatically set up. □

It is very clear that the proposition (2.5) of [Crâşmăreanu and Hreţcanu (2008)] also holds good in our case, i.e.,

Proposition 2.3. *The projector operators L_1 and L_2 defined on a Divine Einstein manifold satisfy the conditions:*

$$L_1 + L_2 = \delta^i_j, \quad L_1^2 = L_1, \quad L_2^2 = L_2 \tag{2.21}$$

and

$$G^i_j o L_1 = L_1 o G^i_j = (1-\phi)L_1, \quad G^i_j o L_2 = L_2 o G^i_j = \phi L_2, \tag{2.22}$$

where

$$L_1 = \frac{1}{\sqrt{5}}(\phi\delta^i_j - G^i_j), \text{ and } L_2 = \frac{1}{\sqrt{5}}[(\phi-1)\delta^i_j + G^i_j]. \tag{2.23}$$

Here, for every $x \in E$, the projectors $L_1 \& L_2$ for two complementary distributions D^{L_1} and D^{L_2} respectively are defined to be the systems of C^∞-tensor fields of the type $(1, 1)$ and will be given by the relations;

$$\pi_i(x) : D^{L_1}(x) \to D^{L_1}_i(x), \ \sum_{i=1}^{k} \pi_i = I, \ \pi_i\pi_j = \delta^i_j\pi_i \tag{2.24}$$

and

$$\pi_i(x) : D^{L_2}(x) \to D^{L_2}_i(x), \ \sum_{i=1}^{k} \pi_i = I, \ \pi_i\pi_j = \delta^i_j\pi_i. \tag{2.25}$$

In order to discuss the conformally Divine Einstein manifolds, let us define the conformal Divine structure as follows:

Definition 2.5. A Divine structure G^i_j characterized by (2.11) is said to be conformally well behaved or conformally weighted (i.e., with weight w), if under the conformal transformation (2.3) or (2.6), $\exists$ a real number w such that

$$G^i_j \to \tilde{G}^i_j = \Omega^w G^i_j. \tag{2.26}$$

In case if $w = 0$, the Divine structure will be called conformally invariant.
To rescale the Divine structure G^i_j, just like for the Einstein metric $\tilde{g}_{ij}$, one can have a metric compatible connection $\tilde{\nabla}_i$ which can be operated upon G^i_j as follows:

$$\tilde{\nabla}_k G^i_j = \nabla_k G^i_j + Q^i_{kl} G^l_j - Q^n_{kj} G^i_n, \tag{2.27}$$

where

$$Q^i_{jk} = 2\psi_{(j}\delta^i_{k)} - g_{jk}\psi^i \tag{2.28}$$

and

$$\psi_i = \Omega^{-1}\nabla_i\Omega = \nabla_a(\ln\Omega); \;\; \psi^i = g^{ij}\psi_j. \tag{2.29}$$

In view of equations (2.6) and (2.2), we can now define the conformally Divine Einstein structure as well as conformally Divine Einstein manifolds as below:

Definition 2.6. The conformally Divine Einstein structure is a pair $(\tilde{g}, \tilde{G})$ which verifies the compatibility condition:

$$\tilde{g}_{ir}\tilde{G}^r_j = \tilde{g}_{rj}\tilde{G}^r_i, \tag{2.30}$$

where $\tilde{g}$ and $\tilde{G}$ are the rescaled Einstein metric (2.6) and rescaled Divine structure (2.26) respectively.

The compatibility condition (2.30) can also be expressed as

$$\tilde{g}_{rs}\tilde{G}^r_i\tilde{G}^s_j = \tilde{g}_{rj}\tilde{G}^r_i + \tilde{g}_{ij}. \tag{2.31}$$

With the aid of (2.30) and (2.31), we have

Definition 2.7. An n-dimensional conformal Einstein manifold $(\tilde{E}, \tilde{g})$ satisfying the condition (2.8) is called "conformally Divine Einstein manifold", if it intimates the conformally Divine Einstein structure $(\tilde{g}, \tilde{G})$ revealed by (2.30) or (2.31). Thus, a triple $(\tilde{E}, \tilde{g}, \tilde{G})$ in which a conformal metric $\tilde{g}$ and a conformal Divine structure $\tilde{G}$ compatible with $\tilde{g}$ are involved is known as conformally Divine Einstein manifold.

In addition to conformally Divine Einstein manifold $(\tilde{E}, \tilde{g}, \tilde{G})$, we have the following useful proposition identical to proposition (2.1).

Proposition 2.4. *A conformally Divine Einstein manifold $(\tilde{E}, \tilde{g}, \tilde{G})$ bears the property;*

$$\tilde{G}^n = F_n\tilde{G}^i_j + F_{n-1}\tilde{\delta}^i_j, \tag{2.32}$$

for any integer number $n > 0$. Here $(F_n)_n$ is the well known Fibonacci sequence.

Proof. Making use of the conformal rescalings (2.5), (2.6) and (2.26) and committing the verification of proposition (2.1) in mind, we have,

$$\tilde{G}^3 = 2\tilde{G}^i_j + \tilde{\delta}^i_j$$

which produces

$$\tilde{G}^3 = 2\Omega^w G^i_j + \Omega^2 g_{jk}\Omega^{-2}g^{ik} \Rightarrow \tilde{G}^3 = 2\Omega^w\tilde{G}^i_j + \delta^i_j.$$

In particular, if $w = 0$, the last expression yields

$$\tilde{G}^3 = 2G^i_j + \delta^i_j.$$

Also, suppose that

$$\tilde{G}^n == F_n\tilde{G}^i_j + F_{n-1}\tilde{\delta}^i_j, (\forall n \geq 0).$$

Then, we have

$$\tilde{G}^{n+1} = F_n\tilde{G}^2 + F_{n-1}\tilde{G}^i_j = (F_n + F_{n-1}\tilde{G}^i_j + F_n\tilde{\delta}^i_j,$$

which due to Fibonacci properties evidently yields (2.32). □

Likewise, with the support of conformal rescaling technique and proposition (2.2), one can state the following:

Proposition 2.5. *The conformally Divine Einstein structure $(\tilde{g}, \tilde{G})$ defined for some n-dimensional conformal Einstein manifold $(\tilde{E}, \tilde{g})$ bears the trace property give as;*

$$\textit{trace}\,(\tilde{G}^2) = \textit{trace}\,(\tilde{G}) + n. \tag{2.33}$$

Proposition 2.6. *The conformal projector operators $\tilde{L_1}$ and $\tilde{L_2}$ defined on a conformally Divine Einstein manifold satisfy the conditions:*

$$\tilde{L_1} + \tilde{L_2} = \delta^i_j, \ \ \tilde{L_1}^2 = \tilde{L_1}, \ \ \tilde{L_2}^2 = \tilde{L_2} \tag{2.34}$$

and

$$\tilde{G}^i_j o\tilde{L_1} = \tilde{L_1} o\tilde{G}^i_j = (1-\phi)\tilde{L_1}; \ \ \tilde{G}^i_j o\tilde{L_2} = \tilde{L_2} o\tilde{G}^i_j = \phi\tilde{L_2}, \tag{2.35}$$

where

$$\tilde{L_1} = \frac{1}{\sqrt{5}}(\phi\delta^i_j - \Omega^w G^i_j); \ \ \tilde{L_2} = \frac{1}{\sqrt{5}}[(\phi-1)\delta^i_j + \Omega^w G^i_j]. \tag{2.36}$$

Here for every point $\tilde{x} \in \tilde{E}$, the conformal projectors $\tilde{L}_1$ and $\tilde{L}_2$ for two complementary distributions $D^{\tilde{L_1}}$ and $D^{\tilde{L_2}}$ respectively are defined to be the systems of C^∞-tensor fields of the type $(1,1)$ together with the weight w and will be given by the relations:

$$\tilde{\pi}_i(\tilde{x}) : D^{\tilde{L_1}}(\tilde{x}) \to D^{\tilde{L_1}}_i(\tilde{x}), \ \ \sum_{i=1}^{k} \tilde{\pi}_i = I, \ \ \tilde{\pi}_i\tilde{\pi}_j = \delta^i_j\tilde{\pi}_i, \tag{2.37}$$

and

$$\tilde{\pi}_i(\tilde{x}) : D^{\tilde{L_2}}(\tilde{x}) \to D^{\tilde{L_2}}_i(\tilde{x}), \ \ \sum_{i=1}^{k} \tilde{\pi}_i = I, \ \ \tilde{\pi}_i\tilde{\pi}_j = \delta^i_j\tilde{\pi}_i. \tag{2.38}$$

Now keeping the foregoing digest in mind, in the following section, we attempt to discuss the three popular versions of conformally Divine Einstein equations which may play vitally important role in the study of Einstein's general relativity and theory of gravitation.

3 Conformally Divine Einstein Equations

The cores of Einstein's theory of relativity and gravitation are his field equations. In the present discussion, we shall come across and concisely delineate three popular versions of conformal Einstein equations by inducing conformally Divine Einstein structures to them so that the previously known conformal Einstein equations could possess new look and advance significance.

The first version of conformal Einstein equation is very well known, while the second is less known. But the basic similarity is that each of the two versions is a set of expressions containing the Einstein metric g_{ij} and the conformal factor Ω. Now, from the definition (2.6), its evident that the conformal Divine structure is compatible with conformally rescaled Einstein metric $\tilde{g}_{ij}$. Hence, it seems possible to introduce the conformal Einstein metric in terms of conformally Divine Einstein structure in all those known conformal Einstein equations, which we want to establish in new fashion.

3.1 The First Version of Conformally Divine Einstein Equations

Let us assume that the rescaled metric $\tilde{g}_{ij}$ is an Einstein one so that the equation (2.8) comes true. However, if we use the following conformal rescalings of the Ricci tensor R_{ij} and the curvature scalar R under the transformation $\tilde{g}_{ij} = \Omega^2 g_{ij}$ [Bergman (2004):

$$\tilde{R}_{ij} = R_{ij} + (n-2)\nabla_i\psi_j + g_{ij}\nabla_k\psi^k - (n-2)\psi_i\psi_j + (n-2)g_{ij}\psi_k\psi^k \tag{3.1}$$

and

$$\tilde{R} = \Omega^{-2}[R + 2(n-1)\nabla_k\psi^k + (n-1)(n-2)\psi_k\psi^k], \tag{3.2}$$

where $\psi_i = \nabla_i(\ln \Omega)$ and $\psi^i = \nabla^i(\ln \Omega)$. Then equation (2.8) becomes

$$R_{ij} - \frac{R}{n}g_{ij} + (n-2)\nabla_i\psi_j - \frac{(n-2)}{n}g_{ij}\nabla_k\psi^k - (n-2)\psi_i\psi_j + \\ + \frac{(n-2)}{n}g_{ij}\psi_k\psi^k. \quad (3.3)$$

The expression (3.3) stands for the first well known conformal Einstein equation.
Now, it is very obvious from equation (2.13) and (2.15) that the Einstein metric g_{ij} is G-compatible and hence involving the condition (2.15) in (3.3), we have a new looking conformal equation as follows:

$$R_{ij} - \frac{R}{n}(g_{rs}G_i^r G_j^s - g_{rj}G_i^r) + (n-2)\nabla_i\psi_j - \frac{(n-2)}{n}(g_{rs}G_i^r G_j^s - g_{rj}G_i^r)\times \\ \nabla_k\psi^k - (n-2)\psi_i\psi_j + \frac{(n-2)}{n}(g_{rs}G_i^r G_j^s - g_{rj}G_i^r).\psi_k\psi^k = 0. \quad (3.4)$$

The equation (3.4) must be called the first version of conformally Divine Einstein equation. Here it should be committed in mind that this version of conformally Divine Einstein equation is a set of equations for both the Divine Einstein structure (g, G) and the conformal parameter ψ. Moreover, this version can be used in pursuing the null-infinity and gravitational radiation and is solvable for the pair $((g, G), \psi)$.

3.2 The Second Version of Conformally Divine Einstein Equations

In order to obtain the second version of conformally Divine Einstein equations, we propose the method discussed by Perkins (2006). According to Perkins (2006), one may simply conformally transform the set of Yang and Bach conditions ($Y_{ijk} = 0,\ B_{ij} = 0$) which are equivalent to the vacuum Einstein equations given by;

$$R_{ij} - \frac{1}{2}g_{ij}R + \Lambda g_{ij} = 0, \quad (3.5)$$

where Λ is the cosmological constant.
Here to develop our second version of the conformally Divine Einstein equations, we start with the conformal transformations of Yang and Bach tensors, after that we introduce the G-compatible Einstein metric g_{ij} in both of the transformations and eventually put forward the vanishing of these G-compatible conformally transformed equations.The step by step mathematical procedure is as follows:
The Yang tensor Y_{ijk} under the conformal transformation (2.6) can be expressed in terms of the symmetric connection ∇_i and the well known Weyl tensor C^i_{jkl} as follows [Perkins (2006)]:

$$\tilde{Y}_{ijk} = Y_{ijk} + \psi_l C^l_{ijk} = \nabla_l C^l_{ijk} + \psi_l C^l_{ijk}, \quad (3.6)$$

where the Weyl tensor C^i_{jkl} has the form [Bergman (2004)]:

$$C^i_{jkl} = R^i_{jkl} + \frac{2}{n(n-2)}\left(\delta^i_l R_{[jk]} - \delta^i_k R_{[jl]} - g_{jl}g^{im}R_{[mk]} + g_{jk}g^{im}R_{[ml]} - (n-2)\delta^i_j R_{[lk]}\right) - \\ - \frac{1}{(n-2)}\left(\delta^i_l R_{jk} - \delta^i_k R_{jl} - g_{jl}g^{im}R_{mk} + g_{jk}g^{im}R_{ml}\right) + \frac{R}{(n-1)(n-2)}\left(g_{jk}\delta^i_l - g_{jl}\delta^i_k\right). \quad (3.7)$$

Now, since the Einstein metric is G-compatible, we make use of the following three relations in (3.7)

$$g_{ij} = g_{rs}G_i^r G_j^s - g_{rj}G_i^r, \quad (3.8)$$

$$g^{ij} = g^{rs}G_r^i G_s^j - g^{rj}G_r^i, \quad (3.9)$$

and

$$\delta^i_j = G^2 - G^i_j. \quad (3.10)$$

We obtain a lengthy but straight forward relation as;

$$C^{i\,[Divine]}_{jkl} = R^i_{jkl} + \frac{2}{n(n-2)}\{G^2R_{[jk]} - G^i_jR_{[jk]} - G^2R_{[jl]} + G^i_kR_{[jl]} - \\ - (g_{rs}G^r_jG^s_l - g_{rl}G^r_j)(G^i_rG^m_sg^{rs} - g^{rm}G^i_r)R_{[mk]} + (g_{rs}G^r_jG^s_k - g_{rk}G^r_j)\times \\ (G^i_rG^m_sg^{rs} - g^{rm}G^i_r)R_{[ml]} - (n-2)(G^2 - G^i_j)R_{[lk]}\} - \frac{1}{(n-2)}\{G^2R_{jk} - G^i_jR_{jk} - \\ - G^2R_{jl} + G^i_kR_{jl} - (g_{rs}G^r_jG^s_l -_{rl} G^r_s)(G^i_rG^m_sg^{rs} - g^{rm}G^i_r)R_{mk} + \\ + (g_{rs}G^r_jG^s_k - g_{rk}G^r_kj)(G^i_rG^m_sg^{rs} - g^{rm}G^i_r)R_{ml}\} + \\ + \frac{R}{(n-1)(n-2)}\{(g_{rs}G^r_jG^s_k - g_{rk}G^r_j)(G^2 - G^i_l) - (g_{rs}G^r_jG^s_j - g_{rl}G^r_j)(G^2 - G^i_k)\}. \quad (3.11)$$

We shall evoke this lengthy expression as a Divine-Weyl tensor and will symbolize it by $C^{i\,[Divine]}_{jkl}$. Employing this Divine-Weyl tensor in equation (3.6), we have

$$\tilde{Y}^{[Divine]}_{ijk} = \nabla_iC^{i\,[Divine]}_{jkl} + \psi_iC^{i\,[Divine]}_{jkl}, \quad (3.12)$$

where $\tilde{Y}^{[Divine]}_{ijk}$ will be cited as a conformal Divine Yang tensor and can be calculated easily using (3.12).

Similarly, writing the conformal transformation for the Bach tensor B_{ij} as follows [Perkins (2006)]:

$$\tilde{B}_{ij} = \tilde{\nabla}^k\tilde{\nabla}_l\tilde{C}^l_{ijk} + \frac{1}{2}\tilde{R}^k_l\tilde{C}^l_{ijk} = \Omega^{-2}(\nabla^k\nabla_lC^l_{ijk} + \frac{1}{2}R^k_lC^l_{ijk}) = \Omega^{-2}B_{ij}. \quad (3.13)$$

The involvement of Divine-Weyl tensor (3.11) in equation (3.13) naturally yields

$$\tilde{B}^{[Divine]}_{ij} = \Omega^{-2}(\nabla^k\nabla_lC^{l\,[Divine]}_{ijk} + \frac{1}{2}R^k_lC^{l\,[Divine]}_{ijk}), \quad (3.14)$$

where the symbol $\tilde{B}^{[Divine]}_{ij}$ stands for the Divine Bach tensor.

Eventually, according to Perkins (2006), putting forward the vanishing of (3.12) and (3.14), we have

$$\begin{cases} \tilde{Y}^{[Divine]}_{ijk} = 0, \\ \tilde{B}^{[Divine]}_{ij} = 0. \end{cases} \quad (3.15)$$

Thus we have obtained the second version of conformally Divine Einstein equations by setting the right hand sides of (3.12) and (3.14) equal to zero, i.e.,

$$\begin{cases} \nabla_iC^{i\,[Divine]}_{jkl} + \psi_iC^{i\,[Divine]}_{jkl} = 0, \\ (\nabla^k\nabla_lC^{l\,[Divine]}_{ijk} + \frac{1}{2}R^k_lC^{l\,[Divine]}_{ijk}) = 0. \end{cases} \quad (3.16)$$

3.3 The Third Version of Conformally Divine Einstein Equations

In the modern literature of conformal Einstein's theory, it has been proposed that one can fabricate the conformal Einstein equations without the explicit involvement of conformal parameter ψ [Perkins (2006)]. This fabrication can be done by combining the first version of the conformal Einstein equations with conformal Yang equations of the second version. Here, to develop the third version of conformally Divine Einstein equations, we follow the approach developed by Perkins (2006) and combine the first conformally Divine Einstein equations (3.4) with the conformally Divine Yang equations (3.16) as follows:

$$R_{ij} - \frac{R}{n}(g_{rs}G^r_iG^s_j - g_{rj}G^r_i) + (n-2)\nabla_i\psi_j - \frac{(n-2)}{n}(g_{rs}G^r_iG^s_j - g_{rj}G^r_i)\times \\ \nabla_k\psi^k - (n-2)\psi_i\psi_j + \frac{(n-2)}{n}(g_{rs}G^r_iG^s_j - g_{rj}G^r_i).\psi_k\psi^k = 0. \quad (3.17)$$

and

$$\nabla_i C^{i\,[Divine]}_{jkl} + \psi_i C^{i\,[Divine]}_{jkl} = 0. \tag{3.18}$$

Now, a natural question arises at this end that how to combine the above two conformally Divine Einstein equations? To overcome this difficulty, we shall follow the techniques revealed by Perkins (2006) as below:

At the very first, we may somehow strive for the solution of conformally Divine Einstein equations in terms of the components of gradient ψ_i, which determines them as the functions of Divine-Weyl tensor (3.11). Since the Divine-Weyl tensor is itself a function of Einstein's metric g_{ij} and this metric is itself a function of Divine Einstein structure (g, G). Hence, ψ_i's are then the functions of Divine Einstein structure, i.e.,

$$\psi_i = K_i(g, G). \tag{3.19}$$

In the next step, we should seek for the replacement of ψ_i of equation (3.17) with the K_i so as to obtain equations involving only (g, G) and its derivatives.

The tediousness with this approach is that how to solve all conformally Divine Yang equations for ψ_i? To rectify this issue Perkins (2006) has recommended some elegant devices which will be employed in our future work. The readers, interested to know about these devices are requested to refer Perkins (2006).

4 Concluding Remarks

Here is the brief discussion over some main outcomes of this article revealed in favor of functioning of Divine ratio on conformal Einstein spaces.

a In the section (1), a brief digest on the historical evolution and applications of Divine number has been discussed.

b In the subsection (1.1), we have studied the development of Divine structure and various algebraic as well as geometric features of this beautiful structure. The Golden Riemann structure has also been delineated.

c Section (2) has been the backbone of our research. In this section, initially Einstein and conformal Einstein spaces are defined. Afterward, compatibility conditions of Divine structure with Einstein metric (2.13), (2.15) and (2.17) are mentioned. Further, with the aid of these compatibility conditions, we have discussed two new spaces called Divine Einstein and conformally Divine Einstein spaces. Besides, some propositions regarding trace properties of (g, G), $(\tilde{g}, \tilde{G})$, projector operators on (E, g, G), conformal behavior of (g, G) and conformal projector operators on $(\tilde{E}, \tilde{g}, \tilde{G})$ are also revealed.

d Finally, in the section (3), which is the most crucial part of this article, by making use of compatibility conditions (2.13), (2.15) and (2.17) we have attempted to evolve three new looking versions of conformally Divine Einstein equations. Each of the three versions is lucidly discussed in individual subsections of this section.

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