British Journal of Mathematics & Computer Science
4(11): 1496-1514, 2014

SCIENCEDOMAIN *international*
www.sciencedomain.org

Statistical Inference for a Simple Step-Stress Model Based on Censored Data from the Kumaraswamy Weibull Distribution

H. R. Rezk[1], A. A. EL Helbawy[1*] and G. R. AL Dayian[1]

[1]*Department of Statistics, Faculty of Commerce, AL-Azhar University (Girls' Branch), Cairo, Egypt.*

Original Research Article

Received: 25 October 2013
Accepted: 03 February 2014
Published: 28 March 2014

Abstract

The step-stress accelerated life tests allow increasing the stress levels on test units at fixed time during the experiment. In this paper, accelerated life tests are considered when lifetime of a product follows a Kumaraswamy Weibull distribution. The shape parameter is assumed to be a log linear function of the stress and a cumulative exposure model holds. Based on Type II and Type I censoring, the maximum likelihood estimates are obtained for the unknown parameters. The reliability and hazard rate functions are estimated at usual conditions of stress. In addition, confidence intervals of the estimators are constructed. Optimum test plans are obtained to minimize the generalized asymptotic variance of the maximum likelihood estimators. Monte Carlo simulation is carried out to investigate the precision of the maximum likelihood estimates. An application using real data is used to indicate the properties of the maximum likelihood estimators.

keywords: *Accelerated life tests; simple step-stress; cumulative exposure; type II censoring; type I censoring; confidence intervals; test of hypothesis; Kumaraswmay Weibull distribution; optimum test plans; generalized asymptotic variance; Monte Carlo simulation.*

1 Introduction

In order to obtain highly reliable products long life-spans, time consuming and expensive tests are often required to collect enough failure data. The standard life-testing methods are not appropriate in such situations and to overcome this difficulty accelerated life tests are applied; wherein the test units are run at higher stress levels (which includes temperature, voltage, pressure, vibration, cycling rate, etc.) to cause rapid failures. Accelerated life tests allow the experimenter to apply severe stresses to obtain information on the parameters of the lifetime distributions more quickly

**Corresponding author: aah_elhelbawy@hotmail.com;*

than under normal operating conditions. Such tests can reduce the testing time and save a lot of manpower, material sources and money. The stress can be applied in different ways: Commonly used methods are constant stress, progressive stress and step-stress (see [1,2,3]).

In step-stress accelerated life testing (ALT), the stress for survival units is generally changed to a higher stress level at a predetermined time. This model assumes that the remaining life of a unit depends only on the current cumulative fraction failed and current stress. Moreover, if it is held at the current stress, survivors will continue failing according to the cumulative distribution function (cdf) of that stress but starting at the age corresponding to the previous fraction failed. This model is called the cumulative exposure (CE) model. Some references in the field of the accelerated life testing include [4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22].

[23] constructed a distribution with two shape parameters on (0, 1). Kumaraswamy (Kum) distribution is applicable to many natural phenomena whose outcomes have lower and upper bounds, such as heights of individuals, scores obtained in a test, atmospheric temperatures and hydrological data. Also, Kum distribution could be appropriate in situations where scientists use probability distributions which have infinite lower and or upper bounds to fit data, when in reality the bounds are finite (see [24]). A compound between Kum distribution and any distribution was constructed by [25].

Weibull distribution is one of the most popular models; it has been extensively used for modeling data in reliability, engineering and biological studies. The need for forms of Weibull distribution arises in many applied areas. In this paper, simple step-stress is applied to Kumaraswamy Weibull distribution. The cumulative distribution function (cdf) and the probability density function (pdf) of the Kumaraswamy Weibull (KumW) distribution are obtained as follows:

$$F(t;\,\theta,\beta,\varphi,\lambda) = 1-[1-[1-\exp(-(\lambda t)^{\varphi})]^{\theta}]^{\beta},\quad t>0, \tag{1}$$

and

$$f(t;\,\theta,\beta,\varphi,\lambda) = \theta\beta\varphi\lambda^{\varphi}t^{\varphi-1}\exp(-(\lambda t)^{\varphi})[1-\exp(-(\lambda t)^{\varphi})]^{\theta-1}[1-[1-\exp(-(\lambda t)^{\varphi})]^{\theta}]^{\beta-1},\quad t>0,\ \theta,\beta,\lambda,\varphi>0, \tag{2}$$

where θ, β and φ are the shape parameters, λ is a scale parameter.

It has three shape parameters. These parameters allow for a high degree for flexibility of the KumW distribution. Some special cases can be obtained from KumW distribution such as Kum exponential, Kum Rayleigh, exponentiated Weibull, exponentiated Rayleigh, exponentiated exponential, Weibull, Rayleigh and exponential distributions. It is wide applicable in reliability, engineering and in other areas of research. The KumW is discussed in details in [26].

The reliability function (rf) of KumW and the hazard rate function (hrf) corresponding to (2), can be written, respectively, as follows:

$$R(t;\,\theta,\beta,\varphi,\lambda) = [1-[1-\exp(-(\lambda t)^{\varphi})]^{\theta}]^{\beta},\quad t>0, \tag{3}$$

and

$$h(t;\,\theta,\beta,\varphi,\lambda) = \frac{\theta\beta\varphi\lambda^{\varphi}t^{\varphi-1}\exp(-(\lambda t)^{\varphi})[1-\exp(-(\lambda t)^{\varphi})]^{\theta-1}}{[1-[1-\exp(-(\lambda t)^{\varphi})]^{\theta}]},\quad t>0. \tag{4}$$

The rest of this paper is organized as follows: in Section 2, the k step-stress accelerated life testing is presented. The statistical inference for simple step-stress life testing based on Type II censoring is obtained in Section 3. In Section 4, the statistical inference for simple step-stress life testing based on Type I censoring is discussed.

2 The K Step-Stress Accelerated Life Testing

Assuming k step-stress accelerated life testing, the model of constant stress is considered in the first step. In this model, the lifetime of the unit is affected by a certain level of stress x_1, where x_1 is larger than the usual stress x_u.

In the consecutive steps, other stresses are considered as x_2, x_3, ..., x_k, where $x_u<x_1<x_2<..<x_k$, then the cumulative exposure model reflects the effect of moving from one stress to another one. In the following subsection some basic assumptions are considered.

2.1 Basic Assumptions

1- For any stress x_j, j=1, 2, ..., k, the lifetime distribution is KumW (θ, β, φ, λ) the pdf can be written as follows:

$$f(t_{ij};\beta,\varphi,\lambda,\theta_j)=$$
$$\theta_j\beta\varphi\lambda^{\varphi}t_{ij}{}^{\varphi-1}\exp\left(-\left(\lambda t_{ij}\right)^{\varphi}\right)\left[1-\exp\left(-\left(\lambda t_{ij}\right)^{\varphi}\right)\right]^{\theta_j-1}\left[1-\left[1-\exp\left(-\left(\lambda t_{ij}\right)^{\varphi}\right)\right]^{\theta_j}\right]^{\beta-1}, \quad (5)$$

where t_{ij} is a random variable of time at the step j and r_j is the number of failures at the step j,

$t_{ij} > 0$, β, θ, λ, $\varphi > 0$, j=1, 2, ..., k and i=1, 2, ..., r_j.

2- β, λ, φ are constants with respect to the stress x, and the shape parameter θ is affected by the stress x_j, j=1, 2, ..., k, through the log linear model in the form

$$\theta_j=\exp\left(a+bx_j\right), \quad (6)$$

where a and b are unknown parameters depending on the nature of the unit and the test method.

3- Suppose that, for a particular pattern of stress, units run at stress x_j starting at time τ_{j-1} and reaching to time τ_j , j=1, 2, ..., k, ($\tau_0 = 0$). The behavior of such units is as follows:

At Step 1, the population fraction F_1 (t) of units failing by time τ_1 under constant stress x_1 is

$$F_1(t)=1-\left[1-[1-\exp(-(\lambda t_{i1})^{\varphi})]^{\exp(a+bx_1)}\right]^{\beta}, 0< t<\tau_1,\ a,b,\beta,\varphi,\lambda>0. \quad (7)$$

If F(t) is the population cumulative distribution fraction of units failing under step-stress, then in the first step:

$$F(t)=F_1(t), \quad 0<t<\tau_1, \quad (8)$$

where τ_1 is the time when the stress is raised from x_1 to x_2.

2.2 The Cumulative Exposure Model for the Remaining Steps

When Step 2, starts, units have equivalent age u_1, which have produced the same fraction failed seen at the end of Step 1. In other words the survivors at time τ_1 will be switched to the stress x_2 beginning at the point u_1, which can be determined as the solution of

$$F_2(u_1) = F_1(\tau_1),$$

$$\left(1-\left(1-\exp\left(-(\lambda u_{j-1})^{\varphi}\right)\right)^{\exp(a+bx_j)}\right)^{\beta} = \left(1-\left(1-\exp\left(-\left(\lambda(\Delta_{j-2}+u_{j-2})\right)^{\varphi}\right)\right)^{\exp(a+bx_{j-1})}\right)^{\beta}, \quad (9)$$

where $\Delta_0 = \tau_1 - \tau_0$, $u_0 = 0$ and $\Delta_{j-2} = \tau_{j-1} - \tau_{j-2}$, j=2, 3, ..., k, by solving (9), one obtains

$$\exp\left(-(\lambda u_{j-1})^{\varphi}\right) = 1-\left(1-\exp\left(-\left(\lambda(\Delta_{j-2}+u_{j-2})\right)^{\varphi}\right)\right)^{\exp\left(b(x_{j-1}-x_j)\right)},$$

by taking the logarithm for two sides, it follows that

$$u_{j-1} = \frac{1}{\lambda}\left[-\ln\left[1-\left[1-\exp\left(-\left(\lambda(\Delta_{j-2}+u_{j-2})\right)^{\varphi}\right)\right]^{\exp\left(b(x_{j-1}-x_j)\right)}\right]\right]^{\frac{1}{\varphi}}, \quad (10)$$

the cumulative exposure model for j steps can be written as follows:

$$F(t) = F_j[t-\tau_{j-1}+u_{j-1}], \qquad \tau_{j-1} \le t \le \tau_j,$$

$$F(t) = 1-\left(1-\left(1-\exp\left(-\left(\lambda(t-\tau_{j-1}+u_{j-1})\right)^{\varphi}\right)\right)^{\exp(a+bx_{j-1})}\right)^{\beta}. \quad (11)$$

Substituting u_{j-1} in (11), it is seen that F(t), for a step-stress pattern which consists of segments of the cdf, $F_1, F_2, \ldots, F_k$, can be written in the form:

$$F(t) = \left|\begin{array}{lr} 0 & t \le \tau_0 \\ F_1(t); & \tau_0 \le t \le \tau_1 \\ F_j(t-\tau_{j-1}+u_{j-1}); & \tau_{j-1} \le t \le \tau_j, j = 2,3,\ldots,k-1 \\ F_k(t-\tau_{k-1}+u_{k-1}); & \tau_{k-1} \le t \le \infty \end{array}\right|, \quad (12)$$

and the associated pdf, f(t), has the following form

$$f(t) = \begin{vmatrix} f_1(t); & & \tau_0 \leq t \leq \tau_1 \\ f_j\left(t - \tau_{j-1} + u_{j-1}\right); & \tau_{j-1} \leq t \leq \tau_j, & j = 2,3, \ldots, k-1 \\ & & \\ f_k(t - \tau_{k-1} + u_{k-1}); & & \tau_{k-1} \leq t \leq \infty \\ 0 & & \text{elsewhere} \end{vmatrix}, \quad (13)$$

The maximum likelihood (ML) method is applied to the step-stress model. The pdf for each test is shown by (13), which is the time derivative of the cdf given by (12). The likelihood function is the product of such pdf's evaluated at failure times if complete sampling is used or of observed pdf's of such survival functions evaluated at censoring times when censoring is applied. It is shown that F(t), differs for units with different step-stress patterns. The likelihood function is used to obtain maximum likelihood estimators (MLEs) of the parameters a, b and β.

3 Inference and Optimal Simple Step-Stress Accelerated Life Tests Based on Type II Censoring

The experiment based on Type II censoring step-stress has the following assumptions:

1- There are k levels of stress x_1, x_2, ..., x_k , where $x_1 < x_2 < \ldots < x_k$, are applied such that each unit is initially put under stress x_1.
2- The experiment begins with n units. The stress x_1 is applied at the first step and the result is n_1 failure times t_{i1}, i=1, 2, ..., n_1 of test units are observed. When stress x_2 is applied at the second step, n_2 failure times t_{i2}, i=1, 2, ..., n_2 are observed. Finally, at Step j, stress x_j is applied, n_j failure times t_{ij}, are observed.
3- The test begins at stress level x_1 if the unit doesn't fail till the predetermined n_1 failures, the stress is raised to x_2 and held until n_2 failures. If it doesn't fail, stress is raised to x_3. In general, if the unit doesn't fail until the occurrence of n_{j-1} at stress x_{j-1}, then the stress is raised to x_j at τ_{j-1} , j=2, 3, ..., k, and held n_j failure in case of censored samples, then the test is continued until the occurrence of a predetermined number of failures $\sum_{j=1}^{k} n_j$. Then there are n_c units still survived, at the Step k, the data would be the failure times of (n-n_c) failed units arranged in order and the units which survived beyond $t_{c\,(\tau_{nk}).}$
4- Then, it is shown from the previous points that:
The stress x_{j-1} is raised to x_j at τ_{j-1}, j=2, 3, ..., k, when exactly n_{j-1} failures are observed. It is assumed that the test is continued until all units fail when exactly n_k failures are observed. Then, the failure n_{j-1}, j=2, 3, ..., k is predetermined but τ_{j-1}, j=2, 3, ..., k and t_c are random variables.

The failure time distribution is assumed to be KumW distribution and the shape parameter is shown as a function of the stress through the log linear model. The likelihood function of the experiment is assumed to have the following form:

$$L\left(t_{ij}; i = 1,2, \ldots, n_j,\ j = 1,2, \ldots, k\right) = \left[\prod_{i=1}^{n_1} f_1(t_{i1})\right]\left[\prod_{j=2}^{k} \prod_{i=1}^{n_j} f_j\left(t_{ij} - \tau_{j-1} + u_{j-1}\right)\right] * \left[1 - F_k(t_c - \tau_{k-1} + u_{k-1})\right]^{n_c} \quad . \quad (14)$$

It is shown from (14) that the likelihood function consists of three parts. The first one represents the likelihood of the first step which is the same as the case of constant stress. The second part shows the likelihood function of the (k-1) other stresses. The third part shows the likelihood function of the survived units by time t_c. Considering the cumulative exposure model to relate cdf under step-stress to the cdf under constant stress and using the previous assumptions, it is clear that:

The failure time distribution at j-th step

$$f(t_{ij}-\tau_{j-1}+u_{j-1}) = \beta\varphi\lambda^{\varphi}\exp\left(a+bx_j\right)((t_{ij}-\tau_{j-1}+u_{j-1})^{\varphi-1}\exp\left(-\left(\lambda(t_{ij}-\tau_{j-1}+u_{j-1})\right)^{\varphi}\right)$$
$$*\left[1-\exp\left(-\left(\lambda(t_{ij}-\tau_{j-1}+u_{j-1})\right)^{\varphi}\right)\right]^{\exp(a+bx_j)-1}$$
$$*\left[1-\left[1-\exp\left(-\left(\lambda(t_{ij}-\tau_{j-1}+u_{j-1})\right)^{\varphi}\right)\right]^{\exp(a+bx_j)}\right]^{\beta-1}, \quad (15)$$

where

$$u_{j-1}=\frac{1}{\lambda}\left[-\ln\left[1-\left[1-\exp\left(-\left(\lambda(\Delta_{j-2}+u_{j-2})\right)^{\varphi}\right)\right]^{\exp(b(x_{j-1}-x_j))}\right]\right]^{\frac{1}{\varphi}}.$$

As a special case, let k=2, $\tau_{j-1}=\tau_1$ and $\tau_{n2}=t_c$, it is shown that:

$$F_2(u_1)=F_1(\tau_1), \quad (16)$$

then

$$u_1=\frac{1}{\lambda}\left[-\ln\left[1-[1-\exp(-(\lambda\tau_1)^{\varphi})]^{\exp(b(x_1-x_2))}\right]\right]^{\frac{1}{\varphi}}, \quad (17)$$

hence, the population cumulative fraction of specimens failing in Step 2, by time t is given by:

$$F(t)=F_2[(t-\tau_1)+u_1],\ \tau_1\le t\le\tau_2, \quad (18)$$

$$F(t)=1-\left[1-\left[1-\exp\left(-\left(\lambda(t-\tau_1+u_1)\right)^{\varphi}\right)\right]^{\exp(a+bx_2)}\right]^{\beta}. \quad (19)$$

When k=2 there are two steps only with two levels of stress x_1 and x_2. In this case, the likelihood function has the following form:

$L\left(\beta,\varphi,\lambda,\underline{\theta};\underline{t}\right)=$

$$\prod_{i=1}^{n_1}\left\{\begin{array}{c}\beta\varphi\lambda^{\varphi}exp(a+bx_1)t_{i1}^{\varphi-1}exp\left(-(\lambda t_{i1})^{\phi}\right)\left(1-exp\left(-(\lambda t_{i1})^{\phi}\right)\right)^{\exp(a+bx_1)-1}\\ *\left(1-\left(1-exp\left(-(\lambda t_{i1})^{\phi}\right)\right)^{\exp(a+bx_1)}\right)^{\beta-1}\end{array}\right\}$$

$$\prod_{i=1}^{n_2}\left\{\begin{array}{c}\beta\varphi\lambda^{\varphi}exp(a+bx_2)(t_{i2}-\tau_1+u_1)^{\varphi-1}exp\left(-\left(\lambda(t_{i2}-\tau_1+u_1)\right)^{\varphi}\right)\\ *\left(1-exp\left(-\left(\lambda(t_{i2}-\tau_1+u_1)\right)^{\varphi}\right)\right)^{\exp(a+bx_2)-1}\left(1-\left(1-exp\left(-\left(\lambda(t_{i2}-\tau_1+u_1)\right)^{\varphi}\right)\right)^{\exp(a+bx_2)}\right)^{\beta-1}\end{array}\right\}$$

$$*\left(1-\left(1-exp\left(-\left(\lambda(t_c-\tau_1+u_1)\right)^{\varphi}\right)\right)^{\exp(a+bx_2)}\right)^{\beta(n_c)}. \tag{20}$$

Suppose λ and φ are known, the logarithm of the likelihood function in (20), denoted by ℓ_2 is given by:

$$\begin{aligned}\ell_2 &= \ln L\left(\beta,\varphi,\lambda,\underline{\theta};\underline{t}\right)\\ &= (n-n_c)ln(\beta\varphi)+\varphi(n-n_c)ln\lambda+\sum_{j=1}^{2}n_j\left(a+bx_j\right)+(\varphi-1)\sum_{i=1}^{n_1}lnt_{i1}\\ &\quad-\sum_{i=1}^{n_1}(\lambda t_{i1})^{\varphi}\quad+(\exp(a+bx_1)-1)+\sum_{i=1}^{n_1}ln\left(\frac{H_1(.)}{1-\exp(-(\lambda t_{i1})^{\varphi})}\right)\\ &\quad+(\beta-1)\sum_{i=1}^{n_1}ln\left(1-H_1(.)\right)\ +(\varphi-1)\sum_{i=1}^{n_2}ln\left(H_2(.)\right)-\sum_{i=1}^{n_2}(\lambda(H_2(.)))^{\varphi}\\ &\quad+\ (exp(a+bx_2)-1)\sum_{i=1}^{n_2}ln\left(1-exp(-(\lambda(H_2(.)))^{\varphi})\right)\\ &\quad+(\beta-1)\sum_{i=1}^{n_2}ln\left[1-\left(1-exp(-(\lambda(H_2(.)))^{\varphi})\right)^{exp(a+bx_2)}\right]\\ &\quad+\beta n_c\ln\left[1-\left(1-\exp\left(-\left(\lambda\left(H_3(.)\right)\right)^{\varphi}\right)\right)^{\exp(a+bx_2)}\right.\end{aligned} \tag{21}$$

where

$$H_1(.)=(1-\exp(-(\lambda t_{i1})^{\varphi}))^{exp(a+bx_1)},$$

$$H_2(.)=(t_{i2}-\tau_1+u_1),$$
$$H_3(.)=(t_c-\tau_1+u_1),$$

and

$$H_4(.)=(1-\exp(-(\lambda\tau_1)^{\varphi}))^{exp\left(b(x_1-x_2)\right)}.$$

The first derivatives of the logarithm of the likelihood function (21), with respect to a, b and β are obtained.

Therefore, the MLEs can be obtained by equating the first derivatives of ℓ_2 to zero. As shown they are nonlinear equations, the estimates $\hat{a}_2$, $\hat{b}_2$ and $\hat{\beta}_2$ are numerically obtained using Newton Raphson method. Depending on the invariance property of the MLEs, the MLE of the shape parameter, θ_u , of the KumW distribution at usual stress x_u, can be estimated using the following equation.

$$\hat{\theta}_{2u} = exp\left(\hat{a}_2 + \hat{b}_2 x_{2u}\right), \tag{22}$$

also, the MLE of the rf under the same usual conditions $\hat{R}_{2u}(t_0)$, can be given by

$$\hat{R}_{2u}(t_0) = \left[1 - [1 - \exp(-(\lambda t_0)^{\varphi})]^{\hat{\theta}_{2u}}\right]^{\hat{\beta}_2}, \tag{23}$$

and the MLE of the hrf under the same usual conditions $\hat{h}_{2u}(t_0)$, is given as follows

$$\hat{h}_{2u}(t_0) = \frac{\hat{\theta}_{2u}\hat{\beta}_2 \varphi \lambda^{\varphi} t_0^{\varphi-1} \exp(-(\lambda t_0)^{\varphi})[1-\exp(-(\lambda t_0)^{\varphi})]^{\hat{\theta}_{2u}-1}}{\left[1-[1-\exp(-(\lambda t_0)^{\varphi})]^{\hat{\theta}_{2u}}\right]}, \tag{24}$$

where t_0 is a mission time.

The asymptotic Fisher information matrix can be written as follows:

$$\tilde{I}_2 = -\left[\frac{\partial^2 \ell 2}{\partial \psi_i \partial \psi_j}\right], \; i, j = 1, 2, 3, \tag{25}$$

where $\psi_1 = a$, $\psi_2 = b$, $\psi_3 = \beta$ and the elements of the information matrix (25) are derived .

3.1 The Confidence Intervals Based on Type II Censoring

For large sample size, the MLEs under appropriate regularity conditions are consistent and asymptotically unbiased as well as asymptotically normally distributed. Therefore, the two sided approximate 100(1- α) % confidence intervals for the MLE say, $\hat{w}$ of a population value w can be obtained by $P\left(-z \le \frac{\hat{w}-w}{\sigma_{\hat{w}}} \le z\right) = (1-\alpha)$ where z is the $100\left(1-\frac{\alpha}{2}\right)$th standard normal percentile. The two sided approximate 100(1- α) % confidence intervals for a, b and β will be respectively, as follows:

$$L_w = \hat{w} - z_{\frac{\alpha}{2}}\sigma_{\hat{w}}, \qquad \text{and} \qquad U_w = \hat{w} + z_{\frac{\alpha}{2}}\sigma_{\hat{w}} \;\; , \tag{26}$$

where $\sigma_{\hat{w}}$ is the standard deviation and in this study $\hat{w}$ is $\hat{a}$, $\hat{b}$ or $\hat{\beta}$, respectively (see [27]).

3.2 Optimum Test Plans Based on Type II Censoring

The generalized asymptotic variance (GAV) of the MLE of the model parameters is the reciprocal of the determinant of the asymptotic Fisher information matrix $\tilde{I}_2$ (see [11]).

That is

$$GAV\left(\hat{a}_2, \hat{b}_2, \hat{\beta}_2\right) = \left|\tilde{I}_2\right|^{-1}. \quad (27)$$

Thus, minimization of GAV is equivalent to maximization of the determinant of $\tilde{I}_2$. Newton Raphson method is applied to determine numerically the best choice of the sub sample proportion allocated to each level of stress which minimizes GAV as defined previously. Accordingly, the corresponding optimal numbers of items allocated to each step of stress can be obtained by getting the first partial derivatives of $|\tilde{I}_2|$ with respect to G_1 and G_2.

$$\frac{\partial|\tilde{I}_2|}{\partial G_j}, \; j = 1,2. \quad (28)$$

Then setting (28) equal to zero, where G_j are the sub sample proportion which can be optimally determined by solving them simultaneously and applying Newton Raphson method. The determinant can be obtained as follows

$$\left|\tilde{I}_2\right| = \frac{\partial^2 \ell_2}{\partial a^2}\frac{\partial^2 \ell_2}{\partial b^2}\frac{\partial^2 \ell_2}{\partial \beta^2} - \frac{\partial^2 \ell_2}{\partial a^2}\left(\frac{\partial^2 \ell_2}{\partial b \partial \beta}\right)^2 - \frac{\partial^2 \ell_2}{\partial \beta^2}\left(\frac{\partial^2 \ell_2}{\partial a \partial b}\right)^2 + 2\left[\frac{\partial^2 \ell_2}{\partial a \partial b}\frac{\partial^2 \ell_2}{\partial a \partial \beta}\frac{\partial^2 \ell_2}{\partial b \partial \beta}\right] - \frac{\partial^2 \ell_2}{\partial b^2}\left(\frac{\partial^2 \ell_2}{\partial a \partial \beta}\right)^2. \quad (29)$$

Remark

When r=n all the results obtained for Type II censoring, results reduce to those of the complete sample.

3.3 Numerical Results

This section aims to investigate the precision of the theoretical results of both estimation and optimal design plans on basis of simulated and real data.

3.3.1 Simulation algorithm

- Several data sets are generated from KumW distribution for a combination of the initial parameter values of a, b and β, and for sample sizes 20, 30, 60 and 100 using 1000 replications for each sample size.
- The transformation between uniform distribution and KumW distribution in step j=l is

$$(U1)_{i,1} = 1 - \left[1 - \left[1 - exp\left(-\left(\lambda t_{i,1}\right)^{\varphi}\right)\right]^{exp(a+bx_1)}\right]^{\beta}. \quad (30)$$

- While the transformation between uniform distribution and KumW distribution in step j=2 is

$$(U2)_{i,1} = 1 - \left[1 - \left[1 - exp(M)\right]^{exp(a+bx_2)}\right]^{\beta}, \quad (31)$$

where

$$M = -(\lambda\left(t_{i,2} - \tau - \left(\frac{1}{\lambda}\left(ln\left(1 - \left(1 - exp\left(-\left(\lambda t_{i,1}\right)^{\varphi}\right)\right)^{exp(b(x_1-x_2))}\right)\right)\right)^{\varphi}\right))$$

- The whole sample size n is with initial values of the parameters a=0.5, b=1.5 and β =1.2, given n_1 = 0.4n, n_2 = 0.5n and n_c= 0.ln.
- It is assumed that there are only two different levels of stress (k=2), x_1=l and x_2=1.5, which are higher than the stress at usual condition, x_u=0.5.
- Number of test units is allocated to each level of stress where G_1=0.4, G_2=0.5, r_j =90%(n_j), j=l, 2.
- The initial parameter values of a, b and β are used in this simulation study to generate t_{ij}, j = 1, 2 and i=l, 2, ..., r_j.
- Computer program is used depending on MathCad 14 using Newton Raphson method to solve the derived nonlinear logarithmic likelihood equations simultaneously.
- Once the values of $\hat{a}_2$, $\hat{b}_2$ and $\hat{\beta}_2$ are obtained, the estimates are used to obtain, depending on (22) and the design stress, x_u=0.5, the shape parameter under this stress, θ_u, is estimated as $\hat{\theta}_{2u} = \exp\,(\hat{a}_2 + \hat{b}_2\hat{x}_{2u})$. Also, the rf, the hrf and their relative absolute bias are estimated at different values of mission times under usual conditions using (23) and (24).
- The performance of the $\hat{a}_2$, $\hat{b}_2$ and $\hat{\beta}_2$ has been evaluated through some measurements of accuracy. In order to study the precision and variation of MLEs (E_2), it is convenient to use, the relative absolute bias (RAB_2), the mean square error (ER_2) and the relative error (RE_2).
- The two sided approximate 100(1-α) % confidence intervals for a, b and β will be obtained using (26). The different sample sizes of n=20, 30, 60, 100 are considered.
- The results are displayed in Tables 1-4.

3.3.2 Concluding remarks

- It is clear from Table l that the MLEs (E_2) are very close to the initial values of the parameters as the sample size increases. Also, as shown in the numerical results the RAB_2, ER_2 and RE_2 are decreasing when the sample size is increasing.
  For all sample sizes we noted that:
- $\hat{\beta}_2$ performs better than other estimates.
- $\hat{b}_2$ performs better than $\hat{a}_2$.
- Table 2 indicates that the reliability decreases when the mission time t_0 increases. The results get better in the sense that the aim of an accelerated life testing experiments is to get large number of failures (reduce the reliability) of the device with high reliability. As t_0 increases the RAB_{R2} increases and when the sample size increases, the rf increases. Also, the RAB_{R2} for the rf decreases when the sample size increases. The hrf increases when the mission time t_0 increases and when t_0 increases the RAB_{h2} decreases.
- The two-sided 95% central asymptotic confidence intervals for the parameters of KumW are displayed in Table 3. This table contains the standard error (SE_2), lower bound (L_2), upper bound (U_2) and the length of the intervals. The interval estimate of the parameters becomes narrower as the sample size increases.
  For all sample sizes, it is clear that:
- The length of the interval for β is shorter than the other lengths.
- The length of the interval for b is shorter than the length of the interval for a.
- Optimum test plans are developed numerically; it can be observed from the numerical results presented in Table 4, that the optimum test plans do not allocate the same number of the test units to each stress. Also, Table 4, includes the expected number of items that

must be allocated to each level of stress represented by n_1^*, n_2^* which minimizes the GAV. As indicated from the results, the optimal GAV of the MLE of the model parameters decreased as the sample size n increased.

Table 1. The E_2, RAB_2, ER_2 and RE_2 of the estimates at different sample sizes

n	Parameter	E_2	RAB_2	ER_2	RE_2
20	a	0.3602	0.2796	0.0385	0.1962
	b	1.7204	0.1469	0.0274	0.1655
	β	1.3469	0.1213	0.0192	0.1386
30	a	0.4119	0.1762	0.0367	0.1916
	b	1.6821	0.1214	0.0259	0.1609
	β	1.3072	0.0893	0.0179	0.1334
60	a	0.4520	0.0980	0.0311	0.1764
	b	1.6394	0.0930	0.0197	0.1403
	β	1.2889	0.0741	0.0126	0.1122
100	a	0.4731	0.0538	0.0275	0.1658
	b	1.5618	0.0412	0.0168	0.1296
	β	1.2427	0.0439	0.0093	0.0964

Table 2. The estimated shape parameter, rf and hrf under usual condition at different sample sizes

n	$\hat{\theta}_{2u}$	t_0	$\hat{R}_{2u}(t_0)$	RAB_{R2}	$\hat{h}_{2u}(t_0)$	RAB_{h2}
20	3.3885	0.3	0.9767	0.0049	0.4466	0.2333
		0.5	0.7263	0.0483	2.8473	0.1557
		0.7	0.2932	0.1486	6.2290	0.1319
		1	0.0230	0.3598	10.5384	0.1235
30	3.5007	0.3	0.9802	0.0014	0.3907	0.3621
		0.5	0.7461	0.0223	2.6758	0.0861
		0.7	0.3140	0.0882	5.9897	0.0884
		1	0.0268	0.2544	10.217	0.0892
60	3.5669	0.3	0.9820	0.0004	0.3622	0.0002
		0.5	0.7564	0.0009	2.5883	0.0506
		0.7	0.3250	0.0562	5.8736	0.0673
		1	0.0288	0.1970	10.0678	0.0733
100	3.5043	0.3	0.9813	0.0003	0.3702	0.0222
		0.5	0.7573	0.0076	2.5410	0.0314
		0.7	0.3328	0.0336	5.6925	0.0344
		1	.0320	0.1074	9.7126	0.0354

Table 3. Confidence intervals of the estimates at confidence level 95% at different sample sizes

n	Parameter	E_2	SE_2	L_2	U_2	Length
20	a	0.3602	0.1962	0.2725	0.4479	0.1755
	b	1.7204	0.1655	1.6464	1.7944	0.1481
	β	1.3469	0.1386	1.2818	1.4121	0.1303
30	a	0.4119	0.1916	0.3419	0.4819	0.1399
	b	1.6821	0.1609	1.6233	1.7409	0.1175
	β	1.3072	0.1338	1.2400	1.3378	0.0977
60	a	0.4520	0.3130	0.3712	0.5328	0.1617
	b	1.6394	0.3050	1.5607	1.7181	0.1575
	β	1.2889	0.2722	1.2186	1.3592	0.1406
100	a	0.4731	0.1658	0.4399	0.5063	0.0663
	B	1.5618	0.1296	1.5359	1.5877	0.0518
	β	1.2427	0.0964	1.2234	1.2620	0.0386

Table 4. The results of optimal design of the life test at different sample sizes

n	n_1	n_2	G_1	G_2	n_1^*	n_2^*	GAV
20	8	10	0.2903	0.3629	6	7	0.009642
30	12	15	0.3164	0.3955	9	12	0.000517
60	24	30	0.3257	0.4071	20	24	0.000340
100	40	50	0.3481	0.4351	35	44	0.000030

3.3.3 Application

The main aim of this subsection is to demonstrate how the proposed method can be used in practice. [25] used Kolmogorov-Smirnov goodness of fit test and data points representing failure time. The data were taken from [28]. The data were 30 items (n=30) tested with test stopped after 20 th failure (r=20). It is assumed that k=2, i.e. there are only two different levels of stresses x_1=0.6 and x_2=1, which are higher than the stress at usual conditions, x_u=0.5. The failure times in the first step are [0.0014, 0.0623, 1.3826, 2.0130, 2.5274, 2.8221, 3.1544, 4.9835, 5.5462, 5.8196, 5.8714, 7.4710] and the failure times in the second step are [7.5080, 7.6667, 8.6122, 9.0442, 9.1153, 9.6477, 10.1547, 10.7582].

The initial parameter values of a, b and β used in this application are a=0.5, b =1.5, β =2, λ=2 and φ=2. Once the estimate values of a, b and β are obtained, the estimators are used to estimate θ_u, as $\hat{\theta}_{2u} = exp\left(\hat{a}_2 + \hat{b}_2 x_{2u}\right)$. Letting the design stress, x_u= 0.5. Also, the reliability function is estimated at different values of mission times under usual conditions depending on (22).

Moreover, the precision and variation of MLEs (E_2) are studied through some convenient measures such as the RAB_2, ER_2 and RE_2. These measures are computed for each parameter in Table 5.

The estimated shape parameter, rf, RAB_{R2}, the hrf and RAB_{h2} under usual condition are shown in Table 6. Table 7 and Table 8 indicate confidence intervals of the parameters at confidence levels 95% and 99%. These tables contain the standard error (SE_2), lower bound (L_2), upper bound (U_2)

and the length of the intervals.

The relationship between the stress and the shape parameter is tested through testing the significance of the coefficient b. Hypothesis test is obtained when α=0.05 and with one degree of freedom, assuming the null hypothesis is b=0. It is rejected and the relationship between the level of the stress and the shape parameter exist.

Table 5. The E_2, RAB_2, SE_2 and RE_2 of the estimates

Parameter	E_2	RAB_2	ER_2	RE_2
a	0.1122	0.7766	0.1504	0.7756
b	1.6020	0.0013	0.5913	0.4806
β	1.1218	0.4391	0.7712	0.4391

Table 6. The estimated shape parameter, rf and hrf under usual condition

		$\hat{\theta}_{2u}$ =2.4897		
t_0	0.3	0.5	0.7	1
$\hat{R}_{2u}(t_0)$	0.9745	0.9434	0.9065	0.8468
RAB_{R2}	0.0214	0.0465	0.0732	0.1104
$\hat{h}_{2u}(t_0)$	0.8274	3.0461	5.5813	8.8508
RAB_{h2}	1.2849	0.2364	0.0142	0.0564

Table 7. Confidence intervals of the estimates at confidence level 95%

Parameter	E_2	SE_2	L_2	U_2	Length
a	0.1122	0.7756	0.0000	0.8878	0.8878
b	1.6020	0.4806	0.0621	3.1379	3.0758
β	1.1218	0.4391	0.6346	2.8782	3.5128

Table 8. Confidence intervals of the estimates at confidence level 99%

Parameter	E_2	SE_2	L_2	U_2	Length
a	0.1122	0.7756	0.0000	1.3288	1.3288
b	1.6020	0.4806	0.0000	3.9068	3.9068
β	1.1218	0.4391	0.0000	4.0149	4.0149

4 Inference and Optimal Simple Step-Stress Accelerated Life Tests Based on Type I Censoring

In Type I censoring step-stress, the stress x_{j-1} is raised to x_j at τ_{j-1}, j=2, 3, ..., k. It is assumed that the test is continued until all units fail or until time t_c. The difference between time step-stress and failure step-stress, is that in failure step-stress, n_{j-1}, j=2, 3, ..., k+1, are predetermined but τ_{j-1}, j=2, 3, ..., k and t_c are random variables. On the other hand, τ_{j-1}, and t_c are predetermined in time step-stress and n_{j-1}, j=2, 3, ..., k+1, are random variables.

4.1 The Maximum Likelihood Estimation Based on Type I Censoring when there are 2 Steps of Stress as a Special Case

As a special case, let k=2, τ_1 is the time at which the stress changes from x_1 to x_2 and t_c is the time at which the experiment is terminated (censoring time). The likelihood function of the experiment is considered to have the same form as (21) but τ_{j-1} and t_c are predetermined in time step-stress and n_{j-1}, j=2, 3, ..., k+1 are random variables. Then the maximum likelihood estimates are obtained for the unknown parameters. The reliability and the hazard rate functions are estimated. In addition, confidence intervals of the estimators are constructed. Optimum test plans are obtained to minimize the generalized asymptotic variance of the maximum likelihood estimators.

4.2 Numerical Results

This subsection aims to illustrate the precision of the theoretical results of both estimation and optimal design problems on basis of simulated data.

4.2.1 Simulation algorithm

The same steps of the algorithm in Subsection (3.3) will be considered in this algorithm with the following data:

- The values τ_0 = 0, τ_1 =2 and t_c= 5.5 are given. Once the values of $\hat{a}_1$, $\hat{b}_1$ and $\hat{\beta}_1$ are obtained, the estimates are used to obtain, depending on (22) and the design stress, x_u=0.5, the shape parameter under this stress, θ_u, is estimated as $\hat{\theta}_{1u}$ =exp ($\hat{a}_1$+$\hat{b}_1 x_{1u}$). Also, the reliability function, the hazard rate function and their relative absolute bias are estimated at different values of mission times under usual conditions using (23) and (24).
- The performance of the estimates $\hat{a}_1$, $\hat{b}_1$ and $\hat{\beta}_1$ has been evaluated through some measurements of accuracy. In order to study the precision and variation of MLEs, it is convenient to use the relative absolute bias (RAB_1), the mean square error (ER_1) and the relative error (RE_1). Depending on the same procedure as in Section 3, the numerical results of the experiment are displayed in Tables 9-12.

4.2.2 Concluding remarks

- It is clear from Table 9 that the MLEs (E_1) are very close to the initial values of the parameters as the sample size increases. Also, as shown in the numerical results the RAB_1, ER_1 and RE_1 are decreasing when the sample size is increasing.
  For all sample sizes we noted that:
- $\hat{\beta}_1$ performs better than other estimates.
- $\hat{b}_1$ performs better than $\hat{a}_1$.
- Table 10, indicates that the reliability decreases when the mission time t_0 increases. The results get better in the sense that the aim of an accelerated life testing experiments is to get large number of failures (reduce the reliability) of the device with high reliability. As t_0 increases the RAB_{R1} increases and when sample size increases, the rf increases. Also, the RAB_{R1} for the rf decreases when the sample size increases. The hrf increases when the mission time t_0 increases and when t_0 increases the RAB_{h1} decreases.
- The two-sided 95% central asymptotic confidence intervals for the parameters of KumW

are displayed in Table 11. This table contains the standard error (SE_1), lower bound (L_1), upper bound (U_1) and the length of the intervals. The interval estimate of the parameters becomes narrower as the sample size increases.

- As shown in Section 3, by setting the $\frac{\partial|\bar{I}|}{\partial\tau_1} = 0$, τ_1 and t_c can be optimally determined by solving them simultaneously.
  For all sample sizes, it is clear that:
- The length of the interval for β is shorter than the other lengths.
- The length of the interval for b is shorter than the length of the interval for a.
- optimum test plans are developed numerically. The expected time, τ_1^*, at which the stress changes from x_1 to x_2 and the expected time, t_c^*, at which the experiment is terminated are displayed in Table 12. As indicated from the results, the optimal GAV of the MLE of the model parameters is decreasing as the sample size n is increasing.

Table 9. The E_1, RAB_1, ER_1 and RE_1 of the estimates at different sample sizes

n	Parameter	E_1	RAB_1	ER_1	RE_1
20	a	0.4023	0.1954	0.0293	0.1712
	b	1.3283	0.1145	0.0217	0.1473
	β	1.3544	0.1287	0.0238	0.1543
30	a	0.4283	0.1434	0.0246	0.1568
	b	1.3391	0.1073	0.0185	0.1360
	β	1.3214	0.1012	0.0193	0.1389
60	a	0.4578	0.0844	0.0209	0.1044
	b	1.3816	0.0789	0.0154	0.1241
	β	1.2914	0.0762	0.0138	0.1175
100	a	0.4703	0.0594	0.0095	0.0975
	b	1.4669	0.0227	0.0121	0.1100
	β	1.2285	0.0237	0.0108	0.1039

Table 10. The estimated shape parameter, rf and hrf under usual condition at different samples sizes

n	$\hat{\theta}_{1u}$	t_0	$\hat{R}_{1u}(t_0)$	RAB_{R1}	$\hat{h}_{1u}(t_0)$	RAB_{h1}
20	2.9051	0.3	0.9583	0.0237	0.6961	0.9225
		0.5	0.6605	0.1345	3.2825	0.3324
		0.7	0.2475	0.2814	6.5168	0.1842
		1	0.0184	0.4880	10.6450	0.1348
30	2.9977	0.3	0.9636	0.0184	0.6252	0.7266
		0.5	0.6803	0.1085	3.1206	0.2667
		0.7	0.2646	0.2317	6.3101	0.1466
		1	0.0211	0.4124	10.3766	0.1062
60	3.1538	0.3	0.9704	0.0114	0.5308	0.4658
		0.5	0.7071	0.0734	2.9186	0.1847
		0.7	0.2871	0.1663	6.0886	0.1064
		1	0.0245	0.3164	10.1263	0.0796
100	3.3326	0.3	0.9772	0.0004	0.4289	0.1843
		0.5	0.9406	0.0295	2.3689	0.0711
		0.7	0.3215	0.0664	5.7077	0.0372
		1	0.0325	0.1255	9.6170	0.0253

Table 11. Confidence bounds of the parameters at confidence level 95% at different sample sizes

n	Parameters	E_1	SE_1	L_1	U_1	Length
20	a	0.4023	0.0293	0.3258	0.4788	0.1531
	b	1.3283	0.0217	1.2624	1.3942	0.1318
	β	1.3544	0.0238	1.2854	1.4234	0.1380
30	a	0.4283	0.1712	0.3658	0.4908	0.1250
	b	1.3391	0.1473	1.2853	1.3929	0.1076
	β	1.3214	0.1543	1.2651	1.3777	0.1127
60	a	0.4578	0.1446	0.4205	0.4951	0.0746
	b	1.3816	0.1241	1.3496	1.4136	0.0641
	β	1.2914	0.1175	1.2110	1.3217	0.0607
100	a	0.4703	0.0975	0.4508	0.4898	0.0390
	b	1.4669	0.1100	1.4449	1.4889	0.0440
	β	1.2285	0.1039	1.2077	1.2493	0.0416

Table 12. The results of optimal design of the life test at different sample sizes under Type I censoring in step-stress

n	n_1	n_2	τ_1^*	t_c^*	n_1^*	n_2^*	GAV
20	10	10	2.81	4.82	11	3	0.00728
30	15	15	2.94	4.79	19	7	0.00095
60	30	30	3.06	4.53	37	13	0.000249
100	50	50	3.39	4.10	61	16	0.0000583

Remarks

The results obtained in this paper can be modified to obtain results for sub-models of KumW distribution under Type I and Type II censored samples such as

- The Kum exponential distribution if $\varphi = 1$.
- The Kum Rayleigh distribution if $\varphi = 2$.
- The exponentiated Weibull distribution if $\beta = 1$.
- The exponentiated Rayleigh distribution if $\beta = 1$, $\varphi = 2$.
- The exponentiated exponential distribution if $\beta = \varphi = 1$.
- The Weibull distribution if $\beta = \theta = 1$, see [29].
- The Rayleigh distribution if $\varphi = 2$, $\beta = \theta = 1$.
- The exponential distribution if $\varphi = \beta = \theta = 1$.

Acknowledgements

The authors would like to thank Referees and the Editor for their constructive comments and corrections which led to improvements of an earlier version of this article.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Nelson W. Accelerated Testing Statistical Models, Test Plans and Data Analysis. John Wiley & Sons, Inc; 1990.

[2] Bagdonavicius V, Nikulin M. Accelerated Life Models: Modeling and Statistical Analysis. Chapman & Hall-CRC press, Boca Raton, Florida; 2002.

[3] Abdel-Hamid A, AL-Hussaini E. Progressive stress accelerated life tests under finite mixture models. Metrika. 2007;66:213-231.

[4] Lydersen S, Rausand M. A systematic approach to accelerated life testing. Reliability Engineering. 1987;18:285-293.

[5] Trujillo H, Bonat E. Step-stress tests: Monte Carlo simulation and evaluation. Microelectronics, Reliability. 1989;29:281-282.

[6] Wu J, Yu H. Statistical inference about the shape parameter of the Burr Type XII distribution under the failure censored sampling plan. Applied Mathematics and Computation. 2005;163:443-482.

[7] Wang B. Unbiased estimations for the exponential distribution based on step-stress accelerated life testing data. Applied Mathematics and Computation. 2006; 173:1227-1237.

[8] Balakrishnan N, Xie Q. Exact inference for a simple step-stress model with Type II hybrid censored data from the exponential distribution. Journal of Statistical Planning and Inference. 2007;137:2543-2563.

[9] Abd-ELfattah A, Hassan A, Nassr S. Estimation in step-stress partially accelerated life tests for the Burr Type XII distribution using Type I censoring. Statistical Methodology. 2008;5:502-514.

[10] Balakrishnan N, Han D. Exact inference for a simple step-stress model with competing risks for failure from exponential distribution under Type II censoring. Journal of Statistical Planning and Inference. 2008;138:4172-4186.

[11] Wu S, Lin Y, Chen S. Optimal step-stress test under Type I progressive group censoring with random removals. Journal of Statistical Planning and Inference. 2008;138:17-826.

[12] Abdel Hamid A. Constant partially accelerated life tests for Burr Type XII distribution with progressive Type II censoring. Computational Statistics and Data Analysis. 2009;53:2511-2523.

[13] Abdel-Hamid A, AL-Hussaini E. Estimation in step-stress accelerated life tests for the exponentiated exponential distribution with Type I censoring. Computational Statistics and Data Analysis. 2009;53:1328-1338.

[14] Chenhua Li N. Optimal simple step-stress accelerated life test design for reliability predication. Journal of Statistical Planning and Inference. 2009;139:1799-1808.

[15] Wang B. Testing for the validity of the assumptions in the exponential step-stress accelerated life testing model. Computational Statistics and Data Analysis. 2009; 53:2702-2729.

[16] Han D, Balakrishnan N. Inference for a simple step-stress model with competing risk for failure from the exponential distribution under time constraint. Computational Statistics and Data Analysis. 2010;54:2066-2081.

[17] Wang B. Interval estimation for exponential progressive Type II censored step-stress accelerated life testing. Journal of Statistical Planning and Inference. 2010; 140:2706-2718.

[18] Ismail A. Planning step-stress life tests with Type II censored data. Scientific Research and Essays. 2011;6:4021-4028.

[19] Bakoban R. Estimation in step-stress partially accelerated life tests for the generalized inverted exponential distribution using Type I censoring. American Journal of Scientific research. 2012;3:25-35.

[20] Hassan A, AL-Thobety A. Optimal design of failure step-stress partially accelerated life tests with Type II censored inverted Weibull data. International Journal of Engineering Research and Application. 2012;2:3242-3253.

[21] Ramadan S, Ramadan K. Bayesian simple step-stress acceleration life testing plan under progressive Type I right censoring for exponential life distribution. Modern Applied Science. 2012;6:91-99.

[22] Mitra S, Ganguly A, Samanta D, Kundu D. On the simple step-stress model for two parameter exponential distribution. Statistical Methodology. 2013;15:95-114.

[23] Kumaraswamy P. A generalized probability density function for double bounded random processes. Journal of Hydrology. 1980;46:79-88.

[24] Jones M. Kumaraswamy's distribution: a beta type distribution with some tractability advantages. Statistical Methodology. 2009;6:70-81.

[25] Cordeiro G, Ortega E, Nadarajah S. The Kumaraswamy Weibull distribution with application to failure data. Journal of the Franklin Institute. 2010;347:1399 -1429.

[26] AL-Dayian G, EL-Helbawy A, Rezk H. Statistical Inference for a Simple Constant Stress Model Based on Censored Data from the Kumaraswamy Weibull Distribution. (Submitted for publication). 2014; In press.

[27] Nelson W. Applied Life Data Analysis. John Wiley, New York; 1982.

[28] Murthy D, Xie M, Jiang R. Weibull models. John Wiley & Sons, Inc; 2004.

[29] Khamis I. Comparison between constant and step-stress tests for Weibull models. International Journal of Quality & Reliability Management. 1997;14:74-81.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
www.sciencedomain.org/review-history.php?iid=477&id=6&aid=4133