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A Class of Riemannian Manifolds with Special Form of Curvature Tensor

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Abstract

Riemannian curvature is determined by the Ricci curvature by a special formula to introduce a special class of Riemannian manifolds which called Jawarneh manifold. Some geometric properties of Jawarneh manifold have been derived and a non-trivial example is obtained to prove the existence.

Keywords: Riemannian manifold, flat manifold, Einstein manifold, symmetric manifold, Ricci symmetric manifold, conformally flat manifold, cosmology.

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1 Introduction

It is well known that on a Riemannian manifold (M^n, g) the curvature tensor vanishes for n = 1, and it is proportional to the metric g for n = 2. But for n = 3 the curvature tensor of a Riemannian manifold is proportional to the Ricci tensor. Hence without loss of generality we can define a special type of Riemannian manifolds by considering a Riemannian manifold (M^n, g) (n>2) with non-vanishing Ricci tensor S and its curvature tensor R satisfies the relation;

$$R(X, Y, Z) = k[S(Y, Z)X - S(X, Z)Y], \tag{1.1}$$

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where ka non-zero scalar and,

$$S(X,Y) = g(LX,Y), \tag{1.2}$$

where L being the symmetric endomorphism of the tangent space at each point of the manifold corresponding to the Ricci tensor S. Such a manifold shall be called Jawarneh manifold (after my cost name). In particular if k = 0, the manifold reduces to a flat manifold. This justifies the definition of the manifold defined by (1.1).

It is known [1,2,3,4,5] that on a Riemannian manifold the First Bianchi identity is,

$$R(X,Y)Z + R(Y.Z)X + R(Z,X)Y = 0. \tag{1.3}$$

Let X, Y $\in T_p(M)$ at a point p $\in$ M, and let γ be a plane spanned by X, Y. Then the sectional curvature with respect to the section γ is defined by [6],

$$k(\gamma) = -\frac{R(X,Y,X,Y)}{g(X,X)g(Y,Y)-g(X,Y)^2}\ . \tag{1.4}$$

Sectional curvature k(γ) is uniquely determined by the section γ and is independent of the vectors X and Y in the section. If the sectional curvature k(γ) is constant for all sections γ at each point of M, then M is said to be a space of constant Riemannian curvature and we have [1,2,3,4,5],

$$R(X,Y)Z = h[g(Y,Z)X-g(X,Z)Y], \tag{1.5}$$

where h is a constant and any C^∞ vector fields X,Y,Z on M.

The Weyl projective curvature tensor P of a Riemannian manifold [6] is defined as,

$$P(X,Y,Z) = R(X,Y,Z) + \frac{1}{(n-1)}[S(X,Z)Y - S(Y,Z)X]. \tag{1.6}$$

A Riemannian manifold is called symmetric manifold (respectively Ricci symmetric) manifold if its curvature tensor and Ricci tensor satisfies,

$$(\nabla_X R)(Y,Z)W = 0, \tag{1.7}$$

$$(\nabla_X S)(Y,Z) = 0. \tag{1.8}$$

It is known [1] that in a Riemannian manifold the Ricci tensor is of Codazzi type respectively cyclic parallel if it satisfies,

$$(\nabla_X S)(Y,Z) - (\nabla_Z S)(Y,X) = 0, \tag{1.9}$$

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0. \tag{1.10}$$

It is also known [4] that the conformal curvature tensor C on a Riemannian manifold is defined as,

$$C(X,Y,Z) = R(X,Y,Z) - 1/((n-2))\ [S(Y,Z)X - S(X,Z)Y + g(Y,Z)LX - g(X,Z)LY] + r/(n-1)(n-2)\ [g(Y,Z)X - g(X,Z)Y] \tag{1.11}$$

In section 2 it is shown that Jawarneh manifold is an Einstein manifold, satisfies Bianchi identity and of constant Riemannian curvature which is conformal to a flat manifold. Also it is shown that Jawarneh manifold is a conformally flat manifold, symmetric manifold, and of constant scalar curvature. Further it is shown that in Jawarneh manifold the Ricci tensor is of Codazzi type and cyclic parallel. Finally it is shown that Jawarneh manifold is of constant curvature and therefor it is locally maximally symmetric manifold.

Finally in section 3a non-trivial example is given to prove the existence of Jawarneh manifold.

2 Jawarneh Manifold

We can easily verify that (1.4) satisfied by (1.1), i. e. Jawarneh manifold satisfies Bianchi first identity.

Contracting (1.1) and using the assumption S(X, Y) ≠ 0 we can get,

$$k = \frac{1}{n-1}. \tag{2.1}$$

This fact will be used exactly in obtaining a non-trivial example in the next section.

It is clear from (1.1) and (1.7) that Jawarneh manifold is a Weyl projectively flat manifold. But it is known [1] that the necessary and sufficient condition for (M^n, g) $(n>2)$ to be of constant Riemannian curvature is that the Weyl projective curvature tensor vanishes identically. Thus by hypothesis we can state,

Theorem 2.1) Jawarneh manifold is of constant Riemannian curvature.

It is known [1] that a Riemannian manifold of constant Riemannian curvature is conformally flat manifold. Thus by hypothesis we can state,

Theorem 2.2) Every Jawarneh manifold is conformally flat manifold.

Also it is known [1] that a Riemannian manifold of constant Riemannian curvature is conformal to a flat manifold. Thus by hypothesis we can state,

Theorem 2.3) Jawarneh manifold is conformal to a flat manifold.

From (1.1) and (1.6) we have,

$$\mathrm{a}[\mathrm{S}(\mathrm{Y},\mathrm{Z})\mathrm{X} - \mathrm{S}(\mathrm{X},\mathrm{Z})\mathrm{Y}] = \mathrm{h}[\mathrm{g}(\mathrm{Y},\mathrm{Z})\mathrm{X} - \mathrm{g}(\mathrm{X},\mathrm{Z})\mathrm{Y}]. \tag{2.2}$$

Contracting this equation we get,

$$\mathrm{S}(\mathrm{Y},\mathrm{Z}) = \frac{h}{a}\mathrm{g}(\mathrm{Y},\mathrm{Z}). \tag{2.3}$$

Hence Einstein manifold is a special case of Jawarneh manifold.

Analogously from (1.1) and (1.3) we can have,

$$\mathrm{S}(\mathrm{Y},\mathrm{Z})g(X,W) = S(W,X)g(Y,Z), \tag{2.4}$$

Contracting this equation we get,

$$\mathrm{S}(\mathrm{Y},\mathrm{Z}) = \frac{r}{n}\mathrm{g}(\mathrm{Y},\mathrm{Z}), \tag{2.5}$$

where r denotes the scalar curvature. Thus we can state,

Theorem 2.4) Every Jawarneh manifold is an Einstein manifold.

Taking covariant derivative of (2.3) we get (1.9). This means our manifold is Ricci symmetric manifold.

Using this fact and taking covariant derivative of (1.1) we get (1.8). Thusby hypothesiswe can state,

Theorem 2.5) Every Jawarneh manifold is symmetric manifold.

On a conformally flat Riemannian manifold we have [2,5],

$$(\nabla_X S)(Y,Z) - (\nabla_Z S)(X,Y) = \frac{1}{2(n-1)}[dr(Y)g(X,Z) - dr(Z)g(X,Y)]. \tag{2.6}$$

By virtue of Theorem (2.4) we have from (2.6),

$$dr(Y)g(X,Z) = dr(Z)g(X,Y). \tag{2.7}$$

This gives,

$$dr(X) = 0. \tag{2.8}$$

Thus by hypothesis we can state,

Theorem 2.6) In Jawarneh manifold the scalar curvature is constant.

Now taking covariant derivative of (1.1) and contracting the result we can have,

$$(\nabla_X S)(Y,Z) = dr(X)g(Y,Z). \tag{2.9}$$

In consequence of (2.7), (2.9) and (2.10) and by hypothesis we can state,

Theorem 2.7) Jawarneh manifold is of Codazzi type Ricci tensor.

Also by virtue of (2.9), (2.7) and (1.11) and by hypothesis we can state,

Theorem 2.8) In Jawarneh manifolds the Ricci tensor is cyclic parallel.

Using theorem (2.2) on (1.12) we get,

$$R(X,Y,Z) = \frac{1}{(n-2)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)LX - g(X,Z)LY]$$
$$-r/(n-1)(n-2)[\mathrm{g(X,Z)Y} - \mathrm{g(Y,Z)X}]. \tag{2.10}$$

In consequence of (2.4) and (2.10) we have,

$$R(X,Y,Z) = r/(n-1)(n-2)[\mathrm{g(Y,Z)X} - \mathrm{g(X,Z)Y}]. \tag{2.11}$$

Thus by hypothesis we can state,

Theorem 2.9) Jawarneh manifold is a manifold of constant curvature.

Spaces of constant curvature are very significant and well known in cosmology. Every space of constant curvature is locally maximally symmetric, i.e. it has n(n+1)/2 number of local isometries [7]. Thus by hypothesis we can state,

Theorem 2.10) Jawarneh manifold is locally maximally symmetric manifold.

Further it is to be noted that cosmological solutions of Einstein equations which contain a three dimensional Space-like surface of a constant curvature are the Robertson-Walker metrics, while a Four dimensional space of constant curvature is the de Sitter model of the universe. De Sitter model possess a three dimensional space of constant curvature and thus belongs to Robertson-Walker metrics.

3 Example of Jawarneh Manifold

Let us consider R^4 endowed with the Riemannian metric [6],

$$d^2 = g_{ij}dx^i dx^j = (x^4)^{\frac{4}{3}}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2 \, , \quad (3.1)$$

where i, j = 1,2,3,4. Then it is known [6] that the only non-vanishing Ricci tensors and the curvature tensors are,

$$\Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = \frac{2}{3x^4} \; ; \; \Gamma^4_{11} = \Gamma^4_{22} = \Gamma^4_{33} = \frac{-2}{3(x^4)^{\frac{1}{3}}} \, ,$$

$$R_{1441} = R_{2442} = R_{3443} = \frac{-2}{9(x^4)^{\frac{2}{3}}} \, , \quad (3.2)$$

$$S_{11} = S_{22} = S_{33} = \frac{-2}{9(x^4)^{\frac{2}{3}}}; \; S_{44} = \frac{-2}{3(x^4)^2} \, , \quad (3.3)$$

And the scalar curvature r = $\frac{-4}{3(x^4)^2}$.

To verify the definition by (1.1) we have to verify the following relations:

$$R_{1441} = a[S_{44}g_{11} + S_{14}g_{14}], \quad (3.4)$$

$$R_{2442} = a[S_{44}g_{22} + S_{24}g_{24}], \quad (3.5)$$

$$R_{3443} = a[S_{44}g_{33} + S_{34}g_{34}]. \quad (3.6)$$

Let a= $\frac{1}{n-2}$= $\frac{1}{3}$ and using (3.2), (3.3) on (3.4) we get,

R.H.S. = $a[S_{44}g_{11} + S_{14}g_{14}]$ = $\frac{1}{3}[\frac{-2}{3(x^4)^2}(x^4)^{\frac{4}{3}} + 0]$
= $\frac{-2}{9(x^4)^{\frac{2}{3}}}$= L.H.S.

Similarly we can show (3.5) and (3.6) are true, whereas the other cases are trivially true. Hence R^4 along with the metric g defined by (3.1) is Jawarneh manifold. Thus we can state,

Theorem 3.1) A Riemannian manifold (M^4,g) endowed with the metric (3.1) is Jawarneh manifold with non-constant scalar curvature.

4 Conclusion

The aim of studying geometry is trying to understand what curvature is and what it tells us about a space. In particular spaces of constant curvature provide the simplest examples of curvature in its exact form. They can provide us with a launching point to understand what types of spaces we should expect from positive curvature, negative curvature and zero curvature. Constant curvature has played an important role in the history of geometry since spaces with constant curvature provided the first examples of non-Euclidian geometries, putting an end of proving Euclid's parallel axiom from the other geometric axioms. The study of spaces of constant curvature is made very manageable because of a few simplifying properties these spaces have. The first such property is the fact that normal neighborhoods in spaces of the same dimension and the same constant curvature are isometric. So any space with constant curvature is locally the same as any other space with the same curvature. The second distinguishing property is that any complete space of constant curvature is a quotient of either S^n; R^nor H^n which complete Jawarneh manifold do.

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Competing Interests

Author has declared that no competing interests exist.

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