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A Study on Meromorphic Harmonic Starlike Functions by Using a New Generalized Differential Operator

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Authors' contributions

This work was carried out in collaboration with both authors. Author FMS planned, conducted this study and wrote the first draft of the manuscript. Author HOG participated in to interpretation of the results and supervision process. Both authors read and approved the final manuscript.

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ABSTRACT

In this study, a new generalized differential operator $\mathfrak{D}$ was obtained first.Then this new differential operator was used to introduce a new class $\mathcal{GS}(n)$ of univalent meromorphic harmonic starlike functions exterior to the unit disc $\widetilde{\mathbb{U}} := \{z\,|z| > 1\}$. Especially, coefficient bounds for this class have been examined. This coefficient condition is also necessary for the class $\overline{\mathcal{GS}(n)}$ which is subclass of meromorphic harmonic functions. Furthermore, some other properties such as distortion theorems and extreme points for these classes were obtained and relevant results were given.

Keywords: Harmonic functions; meromorphic; starlike functions.

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1 INTRODUCTION

$f = u + iv$ is a complex harmonic function in a domain $\mathbb{D} \subset \mathbb{C}$ if each of u and v is real continuous harmonic functions in $\mathbb{D}$. In any simply connected domain, f is writen in the form of $h + \overline{g}$ where both h and g are analytic in $\mathbb{D}$ [1].

A necessary and sufficient condition for f to be locally univalent and orientation preserving in $\mathbb{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathbb{D}$ [1]. The harmonic functions in the exterior of the unit disc $\widetilde{\mathbb{U}} := \{z\, |z| > 1\}$ were investigated by Hengartner and Schober in [2], and they were represented by the following equation of (1.1)

$$f(z) = h(z) + \overline{g(z)} + A \log |z|, \qquad (1.1)$$

where

$h(z)$ and $g(z)$ are defined by

$$h(z) = \gamma z + \sum_{k=1}^{\infty} a_k z^{-k} and \; g(z) = \beta z + \sum_{k=1}^{\infty} b_k z^{-k} \qquad (1.2)$$

for $0 \leq |\beta| < |\gamma|$, $A \in \mathbb{C}$ and $z \in \widetilde{\mathbb{U}}$. In addition, different classes of meromorphic harmonic functions have been studied by Jahangiri and Silverman [3], Jahangiri [4] and Murugunsundaramoorthy [5,6]. Since harmonic functions are been used in many fields of sciences, new studies on harmonic functions are still of scientific interest.

In this study, a new operator $\mathfrak{D}$ was defined for meromorphic harmonic functions in $\widetilde{\mathbb{U}}$. The classes $\mathcal{GS}(n)$ and $\overline{\mathcal{GS}(n)}$ were also defined. Some properties of these classes, such as coefficient estimates and a distortion theorem, were then investigated. This new operator $\mathfrak{D}$ is defined as follows:

$$\mathfrak{D}^0 f(z) = f(z)$$

$$\mathfrak{D}^1 f(z) = (\lambda - \alpha) \frac{\overline{\left(z^{\frac{\lambda-\alpha+1}{\lambda-\alpha}} g(z)\right)'}}{z^{\frac{1}{\lambda-\alpha}}} - (\lambda - \alpha) z^{\frac{2\lambda-2\alpha+1}{\lambda-\alpha}} \left(\frac{h(z)}{z^{\frac{\lambda-\alpha+1}{\lambda-\alpha}}} \right)'$$

and for $n = 2, ...,$

$$\mathfrak{D}^n f(z) = \mathfrak{D}(\mathfrak{D}^{n-1} f(z)).$$

Using this new operator,

$$\mathfrak{D}^n f(z) = \gamma z + \sum_{k=1}^{\infty} [k(\lambda - \alpha) + (\lambda - \alpha + 1)]^n a_k z^{-k}$$
$$\overline{+(2\lambda - 2\alpha + 1)^n \beta z + (-1)^n \sum_{k=1}^{\infty} [(k-1)(\lambda - \alpha) - 1]^n b_k z^{-k}}$$

was obtained for $n = 0, 1, ...,$ and $0 \leq (2\lambda - 2\alpha + 1)^n |\beta| < |\gamma|$.

Let $\mathcal{GS}(n)$ show the class of harmonic functions with sense preserving and univalent functions that consist of functions satisfying for $z \in \widetilde{\mathbb{U}}$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$,

$$\Re \left\{ 2 - \frac{\mathfrak{D}^{n+1} f(z)}{\mathfrak{D}^n f(z)} \right\} > 0. \qquad (1.3)$$

Also, let $\overline{\mathcal{GS}(n)}$ be the subclass of $\mathcal{GS}(n)$ which consists of meromorphic harmonic functions of the form of (1.4)

$$f_n(z) = h(z) + \overline{g_n(z)} = -\gamma z - \sum_{k=1}^{\infty} a_k z^{-k} + \overline{\beta z - (-1)^n \sum_{k=1}^{\infty} b_k z^{-k}} \qquad (1.4)$$

where $\gamma > \beta \geq 0, a_k \geq 0, b_k \geq 0$. A necessary and sufficient condition for f functions of the form (1.1) to be starlike in $\widetilde{\mathbb{U}}$ is that

$$\frac{\partial}{\partial\theta} arg(f(re^{i\theta})) = \Re\left\{\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right\} > 0, \qquad z \in \widetilde{\mathbb{U}}. \tag{1.5}$$

for each z, $|z| = r > 1$. This classification (1.5) for harmonic univalent functions was first used by Jahangiri [7].

2 COEFFICIENT INEQUALITIES

In this section, sufficient conditions of coefficient inequalities for $f(z)$ to belongs to the class $\mathcal{GS}(n)$ are obtained.

Theorem 2.1. *If* $f(z) = h(z) + \overline{g(z)}$ *where* $\frac{1}{2} \leq \lambda - \alpha \leq 1$, $\alpha \geq 0, \lambda \geq 0$, $h(z)$ *and* $g(z)$ *are of the form* (1.2) *and the inequality*

$$\sum_{k=1}^{\infty}[k(\lambda-\alpha)+(\lambda-\alpha+1)]^n[k(\lambda-\alpha)+(\lambda-\alpha-1)]|a_k|$$
$$+\sum_{k=2}^{\infty}[(k-1)(\lambda-\alpha)-1]^n[(k-1)(\lambda-\alpha)+1]|b_k|+|b_1| \leq |\gamma|-(2\lambda-2\alpha+1)^n(2\lambda-2\alpha-1)|\beta| \tag{2.1}$$

is satisfied, then $f(z)$ *is univalent, sense preserving and* $f(z) \in \mathcal{GS}(n)$ *in* $\widetilde{\mathbb{U}}$.

Proof. We must show that if the condition (2.1) is satisfied, then $f(z) \in \mathcal{GS}(n)$. Hence, it is sufficient to show that $p_n(z)$ is in the class $\mathcal{GS}(n)$ which is the class of harmonic functions with positive real part.

$$|p_n(z)+1| > |p_n(z)-1|, z \in \widetilde{\mathbb{U}}, \tag{2.2}$$

where

$$p_n(z) = \frac{2\mathfrak{D}^n f(z) - \mathfrak{D}^{n+1} f(z)}{\mathfrak{D}^n f(z)} \tag{2.3}$$

from (2.2) we obtain,

$$\frac{|3\mathfrak{D}^n f(z) - \mathfrak{D}^{n+1} f(z)|}{|\mathfrak{D}^n f(z)|} - \frac{|\mathfrak{D}^n f(z) - \mathfrak{D}^{n+1} f(z)|}{|\mathfrak{D}^n f(z)|} > 0. \tag{2.4}$$

Since

$$|3\mathfrak{D}^n f(z) - \mathfrak{D}^{n+1} f(z)| - |\mathfrak{D}^n f(z) - \mathfrak{D}^{n+1} f(z)|$$
$$= |2\gamma z+\sum_{k=1}^{\infty}[k(\lambda-\alpha)+(\lambda-\alpha+1)]^n[-k(\lambda-\alpha)+(-\lambda+\alpha+2)]a_k z^{-k}+\overline{(2\lambda-2\alpha+1)^n(2-2\lambda+2\alpha)\beta z}$$
$$\overline{+(-1)^n\sum_{k=1}^{\infty}[(k-1)(\lambda-\alpha)-1]^n[(k-1)(\lambda-\alpha)+2]b_k z^{-k}}|$$
$$-|\sum_{k=1}^{\infty}[k(\lambda-\alpha)+(\lambda-\alpha+1)]^n[-k(\lambda-\alpha)-(\lambda-\alpha)]a_k z^{-k}+\overline{(2\lambda-2\alpha+1)^n(-2\lambda+2\alpha)\beta z}$$
$$\overline{+(-1)^n\sum_{k=1}^{\infty}[(k-1)(\lambda-\alpha)-1]^n[(k-1)(\lambda-\alpha)]b_k z^{-k}}|$$

$$\geq 2|\gamma||z|-|\sum_{k=1}^{\infty}[k(\lambda-\alpha)+(\lambda-\alpha+1)]^n[-k(\lambda-\alpha)-(\lambda-\alpha-2)]a_kz^{-k}|-|(2\lambda-2\alpha+1)^n(2-2\lambda+2\alpha)\beta z|-2|b_1||z|^{-1}$$

$$-|\sum_{k=2}^{\infty}[(k-1)(\lambda-\alpha)-1]^n[(k-1)(\lambda-\alpha)+2]b_kz^{-k}|-|\sum_{k=1}^{\infty}[k(\lambda-\alpha)+(\lambda-\alpha+1)]^n[-k(\lambda-\alpha)-(\lambda-\alpha)]a_kz^{-k}|$$

$$-|(2\lambda-2\alpha+1)^n(-2\lambda+2\alpha)\beta z|-|\sum_{k=2}^{\infty}[(k-1)(\lambda-\alpha)-1]^n[(k-1)(\lambda-\alpha)]b_kz^{-k}|$$

$$\geq 2|\gamma||z|-\sum_{k=1}^{\infty}|[k(\lambda-\alpha)+(\lambda-\alpha+1)]^n[-k(\lambda-\alpha)-(\lambda-\alpha-2)]||a_k||z|^{-k}-|(2\lambda-2\alpha+1)^n(2-2\lambda+2\alpha)||\beta||z|$$

$$-\sum_{k=2}^{\infty}|[(k-1)(\lambda-\alpha)-1]^n[(k-1)(\lambda-\alpha)+2]||b_k||z|^{-k}-2|b_1|$$

$$-\sum_{k=1}^{\infty}|[k(\lambda-\alpha)+(\lambda-\alpha+1)]^n[-k(\lambda-\alpha)-(\lambda-\alpha)]||a_k||z|^{-k}$$

$$-|(2\lambda-2\alpha+1)^n(-2\lambda+2\alpha)||\beta||z|-\sum_{k=2}^{\infty}|[(k-1)(\lambda-\alpha)-1]^n[(k-1)(\lambda-\alpha)]||b_k||z|^{-k}$$

$$\geq 2\{|\gamma|-\sum_{k=1}^{\infty}[k(\lambda-\alpha)+(\lambda-\alpha+1)]^n[k(\lambda-\alpha)+(\lambda-\alpha-1)]|a_k|-(2\lambda-2\alpha+1)^n(2\lambda-2\alpha-1)|\beta|$$

$$-|b_1|-\sum_{k=2}^{\infty}[(k-1)(\lambda-\alpha)-1]^n[(k-1)(\lambda-\alpha)+1]|b_k|\}\geq 0. \quad (2.5)$$

So the proof of Theorem 2.1 is complete. □

In the following theorem we show that the sufficient coefficient condition given by (2.1) is also necessary for the family $\overline{\mathcal{GS}(n)}$.

Theorem 2.2. *Let $f_n(z)=h(z)+\overline{g_n(z)}$. Then $f_n(z)\in\overline{\mathcal{GS}(n)}$ if and only if*

$$\sum_{k=1}^{\infty}[k(\lambda-\alpha)+(\lambda-\alpha+1)]^n[k(\lambda-\alpha)+(\lambda-\alpha-1)]a_k$$

$$+\sum_{k=2}^{\infty}[(k-1)(\lambda-\alpha)-1]^n[(k-1)(\lambda-\alpha)+1]b_k+b_1\leq\gamma-(2\lambda-2\alpha+1)^n(2\lambda-2\alpha-1)\beta. \quad (2.6)$$

Proof. Taking into account of Theorem 2.1, we need to prove the "only if" part, since $\overline{\mathcal{GS}(n)}\subset\mathcal{GS}(n)$. Let $f_n(z)\in\overline{\mathcal{GS}(n)}$, and z be a complex number. If $\Re(z)>0$ then $\Re(\frac{1}{z})>0$. Therefore, we obtained that as follows.

$$0<\Re\left\{\frac{\mathfrak{D}^nf(z)}{2\mathfrak{D}^nf(z)-\mathfrak{D}^{n+1}f(z)}\right\}\leq\left|\frac{\mathfrak{D}^nf(z)}{2\mathfrak{D}^nf(z)-\mathfrak{D}^{n+1}f(z)}\right|$$

$$=\left|\frac{-\gamma z-\sum_{k=1}^{\infty}Aa_kz^{-k}+\overline{(2\lambda-2\alpha+1)^n\beta z-(-1)^n\sum_{k=1}^{\infty}Cb_kz^{-k}}}{-\gamma z+\sum_{k=1}^{\infty}ABa_kz^{-k}+\overline{(2\lambda-2\alpha+1)^n(-2\lambda+2\alpha+1)\beta z-(-1)^n\sum_{k=1}^{\infty}CDb_kz^{-k}}}\right|$$

$$\leq \frac{\gamma|z| + \sum_{k=1}^{\infty} Aa_k|z|^{-k} + (2\lambda-2\alpha+1)^n\beta|z| + \sum_{k=2}^{\infty} Cb_k|z|^{-k} + b_1|z|^{-k}}{\gamma|z| - \sum_{k=1}^{\infty} ABa_k|z|^{-k} - (2\lambda-2\alpha+1)^n(2\lambda-2\alpha-1)\beta|z| - \sum_{k=2}^{\infty} CDb_k|z|^{-k} - b_1|z|^{-k}}$$
$$< \frac{\gamma + \sum_{k=1}^{\infty} Aa_k + (2\lambda-2\alpha+1)^n\beta + \sum_{k=2}^{\infty} Cb_k + b_1}{\gamma - \sum_{k=1}^{\infty} ABa_k - (2\lambda-2\alpha+1)^n(2\lambda-2\alpha-1)\beta - \sum_{k=2}^{\infty} CDb_k - b_1}. \tag{2.7}$$

where

$A = [k(\lambda-\alpha)+(\lambda-\alpha+1)]^n, B = [k(\lambda-\alpha)+(\lambda-\alpha-1)], C = [(k-1)(\lambda-\alpha)-1]^n, D = [(k-1)(\lambda-\alpha)+1].$

The inequation (2.7) leads to the following inequality.

$$\sum_{k=1}^{\infty}[k(\lambda-\alpha)+(\lambda-\alpha+1)]^n[k(\lambda-\alpha)+(\lambda-\alpha-1)]a_k+\sum_{k=2}^{\infty}[(k-1)(\lambda-\alpha)-1]^n[(k-1)(\lambda-\alpha)+1]b_k+b_1$$
$$\leq \gamma-(2\lambda-2\alpha+1)^n(2\lambda-2\alpha-1)\beta.$$

So, the proof of the theorem is being completed. □

3 A DISTORTION THEOREM AND EXTREME POINTS

In this section we will obtain distortion bounds and extreme points for functions $f(z) \in \overline{\mathcal{GS}(n)}$ which f_n is defined by (1.4).

Theorem 3.1. *Let the function $f_n(z)$ be in the class $\overline{\mathcal{GS}(n)}$. Then for $0 < |z| = r < 1$, we have*

$$(\gamma-\beta)r-[\gamma-(2\lambda-2\alpha+1)^n(2\lambda-2\alpha-1)\beta]r^{-1} \leq |f_n(z)| \leq (\gamma+\beta)r+[\gamma-(2\lambda-2\alpha+1)^n(2\lambda-2\alpha-1)\beta]r^{-1}. \tag{3.1}$$

Proof. Taking into account of Theorem 2.2 ,for $0 < |z| = r < 1$, we obtain

$$|f_n(z)| = \left|-\gamma z - \sum_{k=1}^{\infty} a_k z^{-k} + \overline{\beta z - (-1)^n \sum_{k=1}^{\infty} b_k z^{-k}}\right|$$
$$\leq \gamma r + \beta r + \sum_{k=1}^{\infty}(a_k+b_k)r^{-k} \leq \gamma r + \beta r + \sum_{k=1}^{\infty}(a_k+b_k)r^{-1}$$
$$\leq \gamma r + \beta r + r^{-1}(\sum_{k=1}^{\infty}[k(\lambda-\alpha)+(\lambda-\alpha+1)]^n[k(\lambda-\alpha)+(\lambda-\alpha-1)]a_k$$
$$+\sum_{k=2}^{\infty}[(k-1)(\lambda-\alpha)-1]^n[(k-1)(\lambda-\alpha)+1]b_k+b_1)$$
$$\leq (\gamma+\beta)r+[\gamma-(2\lambda-2\alpha+1)^n(2\lambda-2\alpha-1)\beta]r^{-1}$$

by the coefficient inequalitiy in (2.6). □

The distortion bounds given in Theorem 3.1 is valid for functions $f_n = h + \overline{g_n}$ which is given in the form (1.4) and it is also known that the bounds is valid for functions of the form $f = h + \overline{g}$ where h and g are given by (1.2) if the coefficient condition (2.1) is satisfied.

The extreme points of closed convex hulls of $\overline{\mathcal{GS}(n)}$ denoted by $clco\overline{\mathcal{GS}(n)}$ were determined in the next theorem.

Theorem 3.2. *Let $f_n = h + \overline{g_n}$ is given by (1.4).Let be $\lambda - \alpha \geq 1$. Then, $f_n \in clco\overline{\mathcal{GS}(n)}$ if and only if it can be expressed as*

$$f_n(z) = \sum_{k=0}^{\infty}[x_k h_{n,k}(z) + y_k g_{n,k}(z)]$$

where $x_k \geq 0, y_k \geq 0$ and $\sum_{k=0}^{\infty}(x_k + y_k) = \gamma$

$$h_{n,0}(z) = -z, \qquad g_{n,0}(z) = -z + \frac{\overline{z}}{(2\lambda - 2\alpha + 1)^n(2\lambda - 2\alpha - 1)}, \qquad g_{n,1}(z) = -z - (-1)^n\overline{z}^{-1},$$

$$h_{n,k}(z) = -z - \frac{1}{[k(\lambda-\alpha) + (\lambda - \alpha + 1)]^n[k(\lambda - \alpha) + (\lambda - \alpha - 1)]}z^{-k}, \qquad k \geq 1 \qquad and$$

$$g_{n,k}(z) = -z - \frac{(-1)^n}{[(k-1)(\lambda-\alpha) - 1]^n[(k-1)(\lambda-\alpha) + 1]}\overline{z}^{-k}, \qquad k \geq 2$$

In particular, the extreme points of $\overline{\mathcal{GS}(n)}$ are $\{h_{n,k}\}$ and $\{g_{n,k}\}$.

Proof. For $\lambda - \alpha > 1$, let

$$f_n(z) = \sum_{k=0}^{\infty}[x_k h_{n,k}(z) + y_k g_{n,k}(z)]$$

where $x_k \geq 0, y_k \geq 0$ and $\sum_{k=0}^{\infty}(x_k + y_k) = \gamma$.

Then we have

$$f_n(z) = \sum_{k=0}^{\infty}[x_k h_{n,k}(z) + y_k g_{n,k}(z)]$$

$$= x_0 h_{n,0}(z) + \sum_{k=1}^{\infty} x_k[-z - \frac{1}{[k(\lambda-\alpha) + (\lambda-\alpha+1)]^n[k(\lambda-\alpha) + (\lambda-\alpha-1)]}z^{-k}]$$

$$+y_0 g_{n,0}(z) + y_1 g_{n,1}(z) + \sum_{k=2}^{\infty} y_k[-z - \frac{(-1)^n}{[(k-1)(\lambda-\alpha) - 1]^n[(k-1)(\lambda-\alpha)+1]}\overline{z}^{-k}]$$

$$= -\sum_{k=0}^{\infty}(x_k + y_k)z - \sum_{k=1}^{\infty}\frac{x_k}{[k(\lambda-\alpha) + (\lambda-\alpha+1)]^n[k(\lambda-\alpha) + (\lambda-\alpha-1)]}z^{-k}$$

$$+\frac{y_0}{[(2\lambda - 2\alpha + 1)^n(2\lambda - 2\alpha - 1)]}\overline{z} - (-1)^n y_1\overline{z}^{-1} - (-1)^n\sum_{k=2}^{\infty}\frac{y_k}{[(k-1)(\lambda-\alpha) - 1]^n[(k-1)(\lambda-\alpha)+1]}\overline{z}^{-k}]$$

$$= -\gamma z - \sum_{k=1}^{\infty}\frac{x_k}{[k(\lambda-\alpha) + (\lambda-\alpha+1)]^n[k(\lambda-\alpha) + (\lambda-\alpha-1)]}z^{-k}$$

$$+\frac{y_0}{[(2\lambda - 2\alpha + 1)^n(2\lambda - 2\alpha - 1)]}\overline{z} - (-1)^n y_1\overline{z}^{-1} - (-1)^n\sum_{k=2}^{\infty}\frac{y_k}{[(k-1)(\lambda-\alpha) - 1]^n[(k-1)(\lambda-\alpha)+1]}\overline{z}^{-k}.$$

Since

$$\sum_{k=1}^{\infty}[k(\lambda-\alpha) + (\lambda-\alpha+1)]^n[k(\lambda-\alpha) + (\lambda-\alpha-1)]\frac{x_k}{[k(\lambda-\alpha) + (\lambda-\alpha+1)]^n[k(\lambda-\alpha) + (\lambda-\alpha-1)]}$$

$$+\sum_{k=2}^{\infty}[(k-1)(\lambda-\alpha) - 1]^n[(k-1)(\lambda-\alpha)+1]\frac{y_k}{[(k-1)(\lambda-\alpha) - 1]^n[(k-1)(\lambda-\alpha)+1]} + y_1$$

$$= \left(\sum_{k=1}^{\infty} x_k + y_1 + \sum_{k=2}^{\infty} y_k\right)$$

$$= \gamma - y_0 - x_0 \leq \gamma - [(2\lambda - 2\alpha + 1)^n (2\lambda - 2\alpha - 1)] \frac{y_0}{[(2\lambda - 2\alpha + 1)^n (2\lambda - 2\alpha - 1)]}$$

by Theorem (2.2), so $f_n(z) \in clco\overline{\mathcal{GS}(n)}$. Conversely, suppose that $f_n(z) \in clco\overline{\mathcal{GS}(n)}$, then we may write

$$f_n(z) = h(z) + \overline{g_n(z)} = -\gamma z - \sum_{k=1}^{\infty} a_k z^{-k} + \overline{\beta z - (-1)^n \sum_{k=1}^{\infty} b_k z^{-k}}$$

where $\gamma > \beta \geq 0, a_k \geq 0, b_k \geq 0$. We set

$$a_k = \frac{x_k}{[k(\lambda - \alpha) + (\lambda - \alpha + 1)]^n [k(\lambda - \alpha) + (\lambda - \alpha - 1)]}, k = 1, 2, ...,$$

$$\beta = \frac{y_0}{[(2\lambda - 2\alpha + 1)^n (2\lambda - 2\alpha - 1)]}, \qquad b_1 = y_1,$$

$$b_k = \frac{y_k}{[(k-1)(\lambda - \alpha) - 1]^n [(k-1)(\lambda - \alpha) + 1]}, k = 2, ...$$

Hence we obtain

$$f_n(z) = h(z) + \overline{g_n(z)} = -\gamma z - \sum_{k=1}^{\infty} a_k z^{-k} + \overline{\beta z - (-1)^n \sum_{k=1}^{\infty} b_k z^{-k}}$$

$$= -\sum_{k=0}^{\infty} (x_k + y_k) z - \sum_{k=1}^{\infty} \frac{x_k}{[k(\lambda - \alpha) + (\lambda - \alpha + 1)]^n [k(\lambda - \alpha) + (\lambda - \alpha - 1)]} z^{-k}$$

$$+ \frac{y_0}{[(2\lambda - 2\alpha + 1)^n (2\lambda - 2\alpha - 1)]} \overline{z} - (-1)^n y_1 \overline{z}^{-1} - \sum_{k=2}^{\infty} \frac{(-1)^n y_k}{[(k-1)(\lambda - \alpha) - 1]^n [(k-1)(\lambda - \alpha) + 1]} \overline{z}^{-k}$$

$$= x_0(-z) + \sum_{k=1}^{\infty} x_k \left[-z - \frac{1}{[k(\lambda - \alpha) + (\lambda - \alpha + 1)]^n [k(\lambda - \alpha) + (\lambda - \alpha - 1)]} z^{-k} \right]$$

$$+ y_0 \left(-z + \frac{\overline{z}}{[(2\lambda - 2\alpha + 1)^n (2\lambda - 2\alpha - 1)]} \right) + y_1 (-z - (-1)^n \overline{z}^{-1})$$

$$+ \sum_{k=2}^{\infty} y_k \left[-z - \frac{(-1)^n}{[(k-1)(\lambda - \alpha) - 1]^n [(k-1)(\lambda - \alpha) + 1]} \overline{z}^{-k} \right].$$

Consequently we obtain following relation as required

$$f_n(z) = \sum_{k=0}^{\infty} [x_k h_{n,k}(z) + y_k g_{n,k}(z)].$$

For $\lambda - \alpha = 1$, we obtain the results of Theorem (6) in [8]. So we avoid giving proof of Theorem (3.2), for this case.

This completes the proof of Theorem (3.2). □

4 CONCLUSIONS

Some new classes of meromorphic harmonic starlike functions involving a new generalized differential operator exterior to the unit disc $\widetilde{\mathbb{U}} := \{z \ |z| > 1\}$ were introduced. Some characteristic properties of these classes are also investigated. Furthermore, For $\lambda = 1$ and $\alpha = 0$ we obtain the results in [8]. Also for $n = 0$ coincides with that given by [3]. Therefore, this work may be considered as a generalization form of earlier works.

COMPETING INTERESTS

The authors declare that no competing interests exist.

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