Hindawi
Advances in Mathematical Physics
Volume 2022, Article ID 1724221, 10 pages
https://doi.org/10.1155/2022/1724221

Research Article

Coupled Fixed Point Theorem for the Generalized Langevin Equation with Four-Point and Strip Conditions

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Received 8 August 2022; Revised 12 November 2022; Accepted 14 November 2022; Published 23 November 2022

Academic Editor: Mohammad Alomari

By considering a metric space with partially ordered sets, we employ the coupled fixed point type to scrutinize the uniqueness theory for the Langevin equation that included two generalized orders. We analyze our problem with four-point and strip conditions. The description of the rigid plate bounded by a Newtonian fluid is provided as an application of our results. The exact solution of this problem and approximate solutions are compared.

1. Introduction

The fractional calculus concept is not absolutely intuitive, where it has no clear geometrical interpretation. Several distinct forms have appeared, to the point that the necessity for order has developed in the field [1, 2]. The variety of potential implementations is even more difficult. One has to think closely about what the inserting of fractional derivatives in the model can provide. Fractional derivatives are generally inserted for modeling processes of mass transport, optics, diffusion, etc. [3, 4]. With the inserting of these derivatives, fractional-order models have described its advantages when modeling supercapacitor capacitances [5] and controllers for temperature [6], DC motors [7], or RC, LC, and RLC electric circuits [8]. Through the present paper, we deem the generalized Langevin differential equation of two generalized different orders:

$$ {}^{c}D^{\beta}\left({}^{c}D^{\alpha}+\lambda\right)x(t)=f(t,x(t)),\ 0\le t\le 1, \tag{1}$$

where $\lambda\in\mathbb{R}$, ${}^{c}D^{\beta}$, and ${}^{c}D^{\alpha}$ are the Caputo generalized derivatives with $0<\alpha\le 1$ and $1<\beta\le 2$, and the continuously differentiable function $f:[0,1]\times\mathbb{R}\longrightarrow\mathbb{R}$ is given. This equation is supplemented with the strip and four-point conditions:

$$\begin{aligned} x(0)&=0,\\ {}^{c}D^{\alpha}x(0)&=0,\\ x(\xi)+x(1)&=\eta+\frac{1}{\Gamma(\gamma)}\int_0^{\eta}(\eta-s)^{\gamma-1}x(s)ds, \end{aligned} \tag{2}$$

where $\gamma>0$ and $0<\eta,\xi<1$. The third boundary condition, which appears as a linear combination of nonlocal point and Riemann-Liouville fractional integral condition on an arbitrary segment $(0,\eta)\subset[0,1]$, can be explained as a gathering of the values of the obscure function at local point 1, and nonlocal point $\xi\in(0,1))$ is proportionate to the strip contribution of the unbeknown function. The investigation of the fractional Langevin equation (1) together with four-point and strip conditions (2) makes our problem new especially when applying coupled immutable point type in the case of an existing mixed monotone mapping.

The main goal of our research is to investigate the solution uniqueness for our case ((1) and (2)) by virtue of applying coupled immutable point type in a metric space with partially ordered set in the case of an existing mixed monotone mapping. It is worth pointing out, as far as we know, that no contribution till now studied the uniqueness of solution for the generalized Langevin differential equation (1) by using coupled immutable point type in a metric space with partially ordered set in the case of an existing mixed monotone mapping except Fazli and Nieto [9]. In a metric space with partially ordered set, the coupled immutable point type in the case of an existing mixed monotone mapping was established at first by Bhaskar and Lakshmikantham [10] and extended by many authors, for instance, [11–13]. The prominence of this process rises from the reality that it is a deductive process that accords convergent sequences to the unique solution of our case ((1) and (2)).

The Brownian motion exceedingly draws through the Langevin equation when the random fluctuation force is submitted to be white noise. If the random oscillation force is not white noise, the object motion is depicted by the generalized Langevin equation [14]. Overall, the ordinary differential equations cannot precisely characterize experimental data and area measurement; as an alternative approach, fractional-order differential equation models are extremely used today [15–17].

The generalized Langevin equation is a substantial differential equation in applied mathematics, physics, and other areas of science and engineering. It has been developed and presented by Mainradi and Pironi [18]. With multipoint and multistrip boundary conditions, [19–21] investigated some properties and results to the solution of fractional Langevin equation. The uniqueness of solution and other properties for boundary value problems of the generalized Langevin equation have drawn a plentiful attention from diversified contributors within the previous decades, check for epitome [22–24] and the spacious roster of references presented therein. Analytical expressions of the correlation functions have been obtained using the two fluctuation-dissipation theorems and fractional calculus approaches.

It is worth mentioning that the Langevin equation is extremely applied to characterize the development of physical phenomena in fluctuating environments. However, for the systems in complex media, the ordinary order to the Langevin equation does not give the true depiction of the dynamics. One of the most important possible generalizations to the Langevin equation is by replacing the derivative of positive integer order by a derivative of fractional order which gives rise to a fractional Langevin equation (see [25] and the references therein).

The Hyers-Ulam-Rassias along with Hyers-Ulam (HU) stability results for fractional Langevin equation has been studied in [26]. An explicit solution to nonhomogeneous fractional delayed Langevin equations has been given in [27]. The stochastic nonlinear fractional Langevin equation with a multiplicative noise has been studied by [28]. The hybrid Sturm-Liouville-Langevin equations with new versions of Caputo fractional derivatives have been investigated in [29].

2. Preliminaries

The authors in [10] inserted the next basic connotations of coupled immutable point type and mixed monotone mapping.

Let $\leq$ be a partial order relation on a nonempty set $\mathbb{S}$ which is reflexive, antisymmetric, and transitive. We refer the pair $(\mathbb{S}, \leq)$ to a partially ordered set. Let s be an element in $\mathbb{S}$, and then s is an upper bound (lower bound) for a subset $U \in \mathbb{S}$ if $u \leq s$ $(s \leq u)$ for each $u \in U$. If there are lower and upper bounds for $\mathbb{S}$, then $(\mathbb{S}, \leq)$ is called bounded partially ordered set. Let r and s be two elements in $(\mathbb{S}, \leq)$, and then r and s are said to be comparable if either $s \leq r$ or $r \leq s$ (or both, in case of $r = s$).

Let us recall the next definitions related to coupled immutable point type and mixed monotone mapping:

Definition 1. Consider that $(\mathbb{S}, \leq)$ is partially ordered and mapping $\mathscr{P} : \mathbb{S} \times \mathbb{S} \longrightarrow \mathbb{S}$. It is called that the mapping $\mathscr{P}$ has mixed monotone property if $\mathscr{P}(p, q)$ is nonincreasing with respect to q and nondecreasing with respect to p for each p and q in $\mathbb{S}$, and this means that

$$q_1 \leq q_2 \text{ such that } q_1, q_2 \in \mathbb{S} \Rightarrow \mathscr{P}(p, q_2) \leq \mathscr{P}(p, q_1), \tag{3}$$

$$p_1 \leq p_2 \text{ such that } p_1, p_2 \in \mathbb{S} \Rightarrow \mathscr{P}(p_1, q) \leq \mathscr{P}(p_2, q). \tag{4}$$

Definition 2. Let $(r, s) \in \mathbb{S} \times \mathbb{S}$, and then (r, s) is called coupled fixed point of $\mathscr{P} : \mathbb{S} \times \mathbb{S} \longrightarrow \mathbb{S}$ if the two relations $\mathscr{P}(r, s) = r$ and $\mathscr{P}(s, r) = s$ are satisfied.

The next coupled immutable point types represent the fundamental outcomes of the contribution [10].

Theorem 3. *Consider the partially ordered $(\mathbb{S}, \leq)$. Postulate that $(\mathbb{S}, d)$ is a complete metric space. Assume that $\mathscr{P} : \mathbb{S} \times \mathbb{S} \longrightarrow \mathbb{S}$ is a continuous mapping that has mixed monotone property on $\mathbb{S}$. Let $\rho \in [0, 1)$ such that*

$$d(\mathscr{P}(r, s), \mathscr{P}(p, q)) \leq \frac{\rho}{2}[d(r, p) + d(s, q)], \forall p \leq r, s \leq q. \tag{5}$$

If $r_0, s_0 \in X$ satisfy the inequalities,

$$\begin{aligned} r_0 &\leq \mathscr{P}(r_0, s_0), \\ \mathscr{P}(s_0, r_0) &\leq s_0. \end{aligned} \tag{6}$$

Assume either

(i) $\mathscr{P}$ is a continuous mapping or

(ii) The set $\mathbb{S}$ satisfies at least one of the following properties:

(a) *If the sequence $r_n \longrightarrow r$ is a nondecreasing, then $r_n \le r$ for all n*

(b) *If the sequence $s_n \longrightarrow s$ is a nonincreasing, then $s \le s_n$ for all n*

Then, there are $r, s \in \mathbb{S}$ that satisfy the equalities

$$\begin{aligned} r &= \mathscr{P}(r, s), \\ s &= \mathscr{P}(s, r). \end{aligned} \tag{7}$$

Let us introduce the next partial order on the space $\mathbb{S} \times \mathbb{S}$

$$(r, s), (p, q) \in \mathbb{S} \times \mathbb{S}, (r, s) \le (p, q) \Leftrightarrow r \le p, q \le s. \tag{8}$$

Theorem 4 (addendum to the presumptions of Theorem 3). *Assume that for each $(r, s), (p, q) \in \mathbb{S} \times \mathbb{S}$, there is a pair $(s, r) \in \mathbb{S} \times \mathbb{S}$ that is comparable to (r, s) and (p, q), and then, $\mathscr{P}$ has a unique coupled immutable pair (r^*, s^*).*

Theorem 5 (addendum to the presumptions of Theorem 3). *Assume that each pair of elements of $\mathbb{S}$ has a lower or an upper bound in $\mathbb{S}$. Then, $r^* = y^*$. Furthermore,*

$$\lim_{n \longrightarrow \infty} \mathscr{P}^n(r_0, y_0) = r^*, \tag{9}$$

where

$$\mathscr{P}^n(r_0, s_0) = \mathscr{P}\left(\mathscr{P}^{n-1}(r_0, s_0), \mathscr{P}^{n-1}(s_0, r_0)\right). \tag{10}$$

Next, let us render sundry famous definitions and identities for fractional calculus. For additional specifics, check [30, 31].

Definition 6. A generalized integral for Riemann-Liouville has the integral form

$$I^\iota h(t) = \int_0^t \frac{(t-\omega)^{\iota-1}}{\Gamma(\iota)} h(\omega) d\omega, \iota > 0, \tag{11}$$

where $\Gamma(\cdot)$ is Gamma function and $h \in C([0,\infty))$, provided that the integral exists.

Definition 7. Suppose $m \in \mathbb{N}$ and ι are positively real with $m-1 < \iota \le m$, the Caputo derivative of $h \in C^m([0,\infty))$ has the integral form

$$^cD^\iota h(t) = \frac{1}{\Gamma(m-\iota)} \int_0^t (t-\omega)^{m-\iota-1} h^{(m)}(\omega) d\omega, \tag{12}$$

provided that the integral exists. We remark that $^cD^\iota c = 0$ where c is constant.

Lemma 8. *Let $m \in \mathbb{N}$ and $m-1 < \iota \le m$. If $h \in C^m([0,\infty))$, then we have*

$$\begin{aligned} I^\iota I^\kappa h(t) &= I^{\iota+\kappa} h(t), \\ I^{\iota c} D^\iota h(t) &= h(t) + c_0 + c_1 t + \cdots + c_{m-1} t^{m-1}, \\ {}^cD^\iota I^\kappa h(t) &= I^{\kappa-\iota} h(t), \kappa \ge \iota. \end{aligned} \tag{13}$$

Lemma 9. *Let n be positive integer and $n-1 < \iota \le n$. Then, we have*

$$\begin{aligned} I^\iota t^\kappa &= \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+\iota+1)} t^{\kappa+\iota}, \kappa > -1, \\ {}^cD^\iota t^\kappa &= \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-\iota+1)} t^{\kappa-\iota}, -1 < \kappa \ne 0, 1, \cdots, n-1, \\ {}^cD^\iota t^\kappa &= 0, \kappa = 0, 1, \cdots, n-1. \end{aligned} \tag{14}$$

Lemma 10. *The generalized Langevin equation (1) with the conditions in (2) has a unique representation of the solution $x(t)$ if and only if the function $x(t)$ is a solution of the integral equation*

$$\begin{aligned} x(t) = {} & \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(x(s), s) ds - \lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s) ds \\ & + \frac{t^{\alpha+1}}{1+\xi^{\alpha+1}} \left\{ \eta + \int_0^\eta \frac{(\eta-s)^{\gamma-1}}{\Gamma(\gamma)} x(s) ds + \lambda \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} x(s) ds \right. \\ & + \lambda \int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} x(s) ds - \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(x(s), s) ds \\ & \left. - \int_0^\xi \frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(x(s), s) ds \right\}. \end{aligned} \tag{15}$$

Proof. In view of Lemmas 8 and 9, we can find

$$^cD^\alpha x(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} h(s) ds + c_0 + c_1 t - \lambda x(t), \tag{16}$$

$$\begin{aligned} x(t) = {} & \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} h(s) ds + \frac{t^\alpha}{\Gamma(\alpha+1)} c_0 + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} c_1 \\ & - \lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s) ds + c_2. \end{aligned} \tag{17}$$

Inserting the condition $x(0) = 0$ in (17) gives $c_2 = 0$, and also inserting the boundary condition $^cDx(0) = 0$ in (16) gives $c_0 = 0$. Using the third boundary condition in (2) gives

$$\begin{aligned} c_1 = {} & \frac{\Gamma\alpha + 2}{1+\xi^{\alpha+1}} \left\{ \eta + \int_0^\eta \frac{(\eta-s)^{\gamma-1}}{\Gamma(\gamma)} u(s) ds + \lambda \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} u(s) ds \right. \\ & + \lambda \int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} u(s) ds - \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(x(s), s) ds \\ & \left. - \int_0^\xi \frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(x(s), s) ds \right\}. \end{aligned} \tag{18}$$

Substituting these values of c_0, c_1, and c_2 in the equation (17), we acquire the desirable results. Conversely, by aiding of the third identity of Lemma 8 and the second identity of Lemma 9, one can see that the solution $x(t)$ satisfies the fractional differential equation (16). Also, it is easy to see that the unique solution $x(t)$ satisfies all boundary conditions in (17). □

3. Main Results

Presume that $(\mathbb{S}, \leq)$ is partially ordered where $\mathbb{S} = C([0,1], \mathbb{R})$; it is evident that $(\mathbb{S}, d)$ is a complete metric space of all continuous functions endowed with the distance

$$d(r, s) = \sup_{t\in[0,1]} |x(t) - y(t)|, r, s \in \mathbb{S}. \tag{19}$$

Distinctly, if the sequence $\{r_m\}_{m\in\mathbb{N}}$ is a nondecreasing in $\mathbb{S}$ and converges to $r \in \mathbb{S}$ and the sequence $\{s_m\}_{m\in\mathbb{N}}$ is a nonincreasing in $\mathbb{S}$ and converges to $s \in \mathbb{S}$, it follows that $r_m \leq r$ and $s \leq s_m$, for all $m \in \mathbb{N}$.

Define the space $(\mathbb{S}\times\mathbb{S}, d)$, and then, it is a complete metric space endowed with the distance

$$d((r, s), (p, q)) = d(r, p) + d(s, q), (r, s), (p, q) \in \mathbb{S}\times\mathbb{S}. \tag{20}$$

Furthermore, the set $(\mathbb{S}\times\mathbb{S}, \leq)$ is partially ordered if we acquaint the next ordered relation in $\mathbb{S}$

$$(r, s) \leq (p, q) \Leftrightarrow r(t) \leq p(t), q(t) \leq s(t), t \in [0, 1]. \tag{21}$$

For any $r, s \in \mathbb{S}$, the functions $\min\{r, s\}$ and $\max\{r, s\}$ are also in $\mathbb{S}$ and are lower bound and upper bound of r and s, respectively. Therefore, for every $(r, s), (p, q) \in \mathbb{S}\times\mathbb{S}$, there exists a $(\max\{r, p\}, \min\{s, q\}) \in \mathbb{S}\times\mathbb{S}$ that is comparable to (r, s) and (p, q).

Previously starting and showing the fundamental outcomes, we insert the next presumptions: Assume that

$(\mathscr{H}_1)$ The function $f : [0, 1]\times\mathbb{R} \longrightarrow \mathbb{R}$ is a jointly continuous

$(\mathscr{H}_2)$ The function f satisfies

$$0 < f(t, x) - f(t, y) \leq Ł(x - y), \forall t \in [0, 1], x, y \in \mathbb{R}, x \geq y, \tag{22}$$

where $Ł > 0$ is the Lipschitz constant

For the sake of computational convenience, we set

$$Q_1 = \frac{2Ł}{\Gamma(\alpha+\beta+1)} + \frac{2}{1+\xi^{\alpha+1}}\left\{\frac{\eta^\gamma}{\Gamma(\gamma+1)} + \frac{\lambda(1+\xi^\alpha)}{\Gamma(\alpha+1)}\right\}, \tag{23}$$

$$Q_2 = \frac{2\lambda}{\Gamma(\alpha+1)} + \frac{2Ł\left(1+\xi^{\alpha+\beta}\right)}{\left(1+\xi^{\alpha+1}\right)\Gamma(\alpha+\beta+1)}, \tag{24}$$

$$\mathcal{Q}_1 = \frac{2Ł}{\Gamma(\alpha+\beta+1)} - \frac{2\lambda}{\Gamma(\alpha+1)} + \frac{2\eta^\gamma}{\left(1+\xi^{\alpha+1}\right)\Gamma(\gamma+1)}, \tag{25}$$

$$\mathcal{Q}_2 = \frac{2}{\left(1+\xi^{\alpha+1}\right)}\left\{\frac{Ł\left(1+\xi^{\alpha+\beta}\right)}{\Gamma(\alpha+\beta+1)} - \frac{\lambda(1+\xi^\alpha)}{\Gamma(\alpha+1)}\right\}. \tag{26}$$

Our mainly investigation is based on the sign of the value of $\lambda \in \mathbb{R}$, so we introduce the results in two ways when $\lambda \geq 0$ and when $\lambda < 0$ as in the following two subsections.

3.1. In the Case of $\lambda \geq 0$. Consider the following two operators:

$$\begin{aligned}(F_1x)(t) = {} & \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(x(s), s)ds \\ & + \frac{t^{\alpha+1}}{1+\xi^{\alpha+1}}\left\{\eta + \int_0^\eta \frac{(\eta-s)^{\gamma-1}}{\Gamma(\gamma)} x(s)ds \right. \\ & + \lambda\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds \\ & \left. + \lambda\int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds\right\},\end{aligned} \tag{27}$$

$$\begin{aligned}(F_2x)(t) = {} & \lambda\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds \\ & + \frac{t^{\alpha+1}}{1+\xi^{\alpha+1}}\left\{\int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(x(s), s)ds \right. \\ & \left. + \int_0^\xi \frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(x(s), s)ds\right\}.\end{aligned} \tag{28}$$

Definition 11. An element $(x_0, y_0) \in X\times X$ is called a coupled lower and upper solution of the boundary value problems (1) and (2) if

$$\begin{aligned} x_0(t) &\leq (F_1x_0)(t) - (F_2y_0)(t), t \in [0, 1], \\ y_0(t) &\geq (F_1y_0)(t) - (F_2x_0)(t), t \in [0, 1]. \end{aligned} \tag{29}$$

Theorem 12. *Through the accompanying presumptions $(\mathscr{H}_1)$ and $(\mathscr{H}_2)$, if the problems (1) and (2) have a coupled upper and lower solutions and $Q < 1$ where $Q = \max\{Q_1, Q_2\}$ and Q_1 and Q_2 are defined as in (23) and (24), respectively, then it has a unique solution in $\mathbb{S}$.*

Proof. Consider the operator

$$F(x, y)(t) = (F_1x)(t) - (F_2y)(t), x, y \in \mathbb{S}, t \in [0, 1], \tag{30}$$

where F_1 and F_2 are defined as in (27) and (28), respectively. The continuity of the operator F comes according to the assumption $(\mathscr{H}_1)$, so $F \in \mathbb{S}$ and it is well defined. Now, let $x_1, x_2 \in \mathbb{S}$ such that $x_1 \le x_2$, and then for each $t \in [0,1]$ and aiding of the assumption $(\mathscr{H}_2)$, we have

$$\begin{aligned}(F_1x_1)(t) - (F_1x_2)(t) &= \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}[f(x_1(s),s) - f(x_2(s),s)]ds \\ &\quad + \frac{t^{\alpha+1}}{1+\xi^{\alpha+1}}\left\{\int_0^\eta \frac{(\eta-s)^{\gamma-1}}{\Gamma(\gamma)}[x_1(s)-x_2(s)]ds\right. \\ &\quad + \lambda\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}[x_1(s)-x_2(s)]ds \\ &\quad \left.+ \lambda\int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}[x_1(s)-x_2(s)]ds\right\} \\ &\le Ł\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}[x_1(s)-x_2(s)]ds \le 0.\end{aligned} \tag{31}$$

Thus, with fix $y \in \mathbb{S}$, we find that

$$F(x_1,y) - F(x_2,y) = F_1x_1 - F_1x_2 \le 0, \tag{32}$$

which leads to $F(x_1,y) \le F(x_2,y)$, and thus, $F(x,y)$ is monotonously nondecreasing in x.

Again, let $y_1, y_2 \in \mathbb{S}$ such that $y_2 \le y_1$, and then for each $t \in [0,1]$ and aiding of the assumption $(\mathscr{H}_2)$, we have

$$\begin{aligned}(F_2y_2)(t) - (F_2y_1)(t) &= \lambda\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[y_2(s)-y_1(s)]ds \\ &\quad + \frac{t^{\alpha+1}}{1+\xi^{\alpha+1}}\left\{\int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}[f(y_2(s),s) - f(y_1(s),s)]ds\right. \\ &\quad \left.+ \int_0^\xi \frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}[f(y_2(s),s)-f(y_1(s),s)]ds\right\} \\ &\le \frac{Łt^{\alpha+1}}{1+\xi^{\alpha+1}}\left\{\int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}[y_2(s)-y_1(s)]ds\right. \\ &\quad \left.+ \int_0^\xi \frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}[y_2(s)-y_1(s)]ds\right\} \le 0.\end{aligned} \tag{33}$$

Thus, with fix $x \in \mathbb{S}$, we find that

$$F(x,y_1) - F(x,y_2) = F_2y_2 - F_2y_1 \le 0, \tag{34}$$

which leads to $F(x,y_1) \le F(x,y_2)$, and thus, $F(x,y)$ is monotonously nonincreasing in y. Therefore, $F(x,y)$ has the mixed monotone property.

For each $x, u \in \mathbb{S}$ with $u \le x$ and $t \in [0,1]$, we have

$$\begin{aligned}&|(F_1x)(t) - (F_1u)(t)| \\ &\le \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}|f(x(s),s) - f(u(s),s)|ds \\ &\quad + \frac{t^{\alpha+1}}{1+\xi^{\alpha+1}}\left\{\int_0^\eta \frac{(\eta-s)^{\gamma-1}}{\Gamma(\gamma)}|x(s)-u(s)|ds\right. \\ &\quad \left.+ \lambda\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|x(s)-u(s)|ds + \lambda\int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}|x(s)-u(s)|ds\right\} \\ &\le Łd(x,u)\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}ds \\ &\quad + \frac{d(x,u)}{1+\xi^{\alpha+1}}\left\{\int_0^\eta \frac{(\eta-s)^{\gamma-1}}{\Gamma(\gamma)}ds + \lambda\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds + \lambda\int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right\} \\ &= \left(\frac{Ł}{\Gamma(\alpha+\beta+1)} + \frac{1}{1+\xi^{\alpha+1}}\left\{\frac{\eta^\gamma}{\Gamma(\gamma+1)} + \frac{\lambda(1+\xi^\alpha)}{\Gamma(\alpha+1)}\right\}\right)d(x,u) \\ &= 12Q_1 d(x,u).\end{aligned} \tag{35}$$

For each $y, v \in \mathbb{S}$ with $y \le v$ and $t \in [0,1]$, we have

$$\begin{aligned}|(F_2y)(t) - (F_2v)(t)| &\le \lambda\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|y(s)-v(s)|ds \\ &\quad + \frac{t^{\alpha+1}}{1+\xi^{\alpha+1}}\left\{\int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}|f(y(s),s) - f(v(s),s)|ds\right. \\ &\quad \left.+ \int_0^\xi \frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}|f(y(s),s)-f(v(s),s)|ds\right\} \\ &\le \left(\frac{\lambda}{\Gamma(\alpha+1)} + \frac{Ł\left(1+\xi^{\alpha+\beta}\right)}{\left(1+\xi^{\alpha+1}\right)\Gamma(\alpha+\beta+1)}\right)d(y,v) \\ &= 12Q_2 d(y,v).\end{aligned} \tag{36}$$

Therefore, for $(x,y), (u,v) \in \mathbb{S}\times\mathbb{S}$, we have

$$\begin{aligned}d(F(x,y), F(u,v)) &= |F(x,y) - F(u,v)| \\ &= |(F_1x)(t) - (F_1u)(t) + (F_2v)(t) - (F_2y)(t)| \\ &\le d(F_1x, F_1u) + d(F_2y, F_2v) \\ &\le 12Q_1 d(x,u) + 12Q_2 d(y,v) \\ &\le 12Qd((x,y),(u,v)).\end{aligned} \tag{37}$$

Now, we have the partially ordered $(\mathbb{S}\times\mathbb{S},\le)$ and the continuous operator $F(x,y)$ which has mixed monotone property. Thus, we emphasize that there exists a coupled lower and upper solution $(x_0(t), y_0(t))$ for the problems (1) and (2) such that $x_0(t) \le F(x_0,y_0)$ and $F(y_0,x_0) \le y_0(t)$. So, for every $(x,y),(u,v) \in \mathbb{S}\times\mathbb{S}$, there exists a $(x_0(t), y_0(t))$ that is comparable to (x,y) and (u,v). These mean that all the assumptions of Theorems 3 and 4 are satisfied. Therefore, F has a unique coupled fixed point in $\mathbb{S}\times\mathbb{S}$;

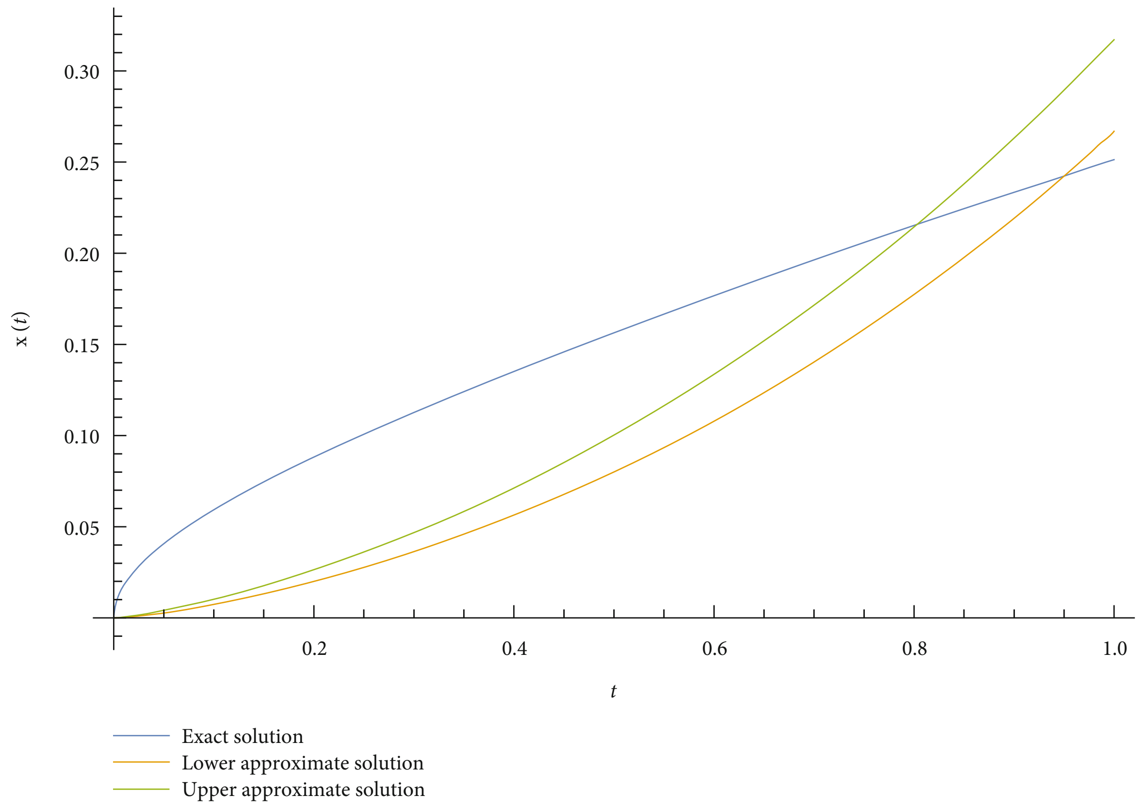

FIGURE 1: Exact solution with approximate solution F^1.

that is, the boundary value problems (1) and (2) have the unique solution $(x^*, y^*) \in [0,1] \times [0,1]$. This coupled solution, according to Theorem 5, can be obtained as

$$\lim_{n \longrightarrow \infty} F^n(x_0(t), y_0(t)) = x^* = y^*. \tag{38}$$

This ends the proof. □

3.2. In the Case of $\lambda < 0$. Consider the following two operators:

$$\begin{aligned}(\mathcal{F}_1 x)(t) &= \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(x(s), s) ds - \lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s) ds \\ &\quad + \frac{t^{\alpha+1}}{1+\xi^{\alpha+1}} \left\{ \eta + \int_0^\eta \frac{(\eta-s)^{\gamma-1}}{\Gamma(\gamma)} x(s) ds \right\}, \\ (\mathcal{F}_2 x)(t) &= \frac{t^{\alpha+1}}{1+\xi^{\alpha+1}} \left\{ \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(x(s), s) ds \right. \\ &\quad + \int_0^\xi \frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(x(s), s) ds - \lambda \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} x(s) ds \\ &\quad \left. - \lambda \int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} x(s) ds \right\}.\end{aligned} \tag{39}$$

Definition 13. The pair $(x_0, y_0) \in \mathbb{S} \times \mathbb{S}$ is called a coupled lower and upper solution of the boundary value problems (1) and (2) if

$$\begin{aligned} x_0(t) &\le (\mathcal{F}_1 x_0)(t) - (\mathcal{F}_2 y_0)(t), t \in [0,1], \\ y_0(t) &\ge (\mathcal{F}_1 y_0)(t) - (\mathcal{F}_2 x_0)(t), t \in [0,1]. \end{aligned} \tag{40}$$

Theorem 14. *Through the accompanying presumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, if the problems (1) and (2) have coupled lower and upper solutions and $\mathcal{Q} < 1$ where $\mathcal{Q} = \max\{\mathcal{Q}_1, \mathcal{Q}_2\}$ and $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are defined as in (25) and (26), respectively, then it has a unique solution in $\mathbb{S}$.*

Proof. Consider the operator

$$\mathcal{F}(x,y)(t) = (\mathcal{F}_1 x)(t) - (\mathcal{F}_2 y)(t), x, y \in \mathbb{S}, t \in [0,1], \tag{41}$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ are defined as in (23) and (24), respectively. The remnant of proof is identical to the proof of the former theorem. □

4. Motion of an Immersed Plate

Consider now the rigid plate of mass m immersed in a Newtonian fluid of infinite extent and connected by a massless

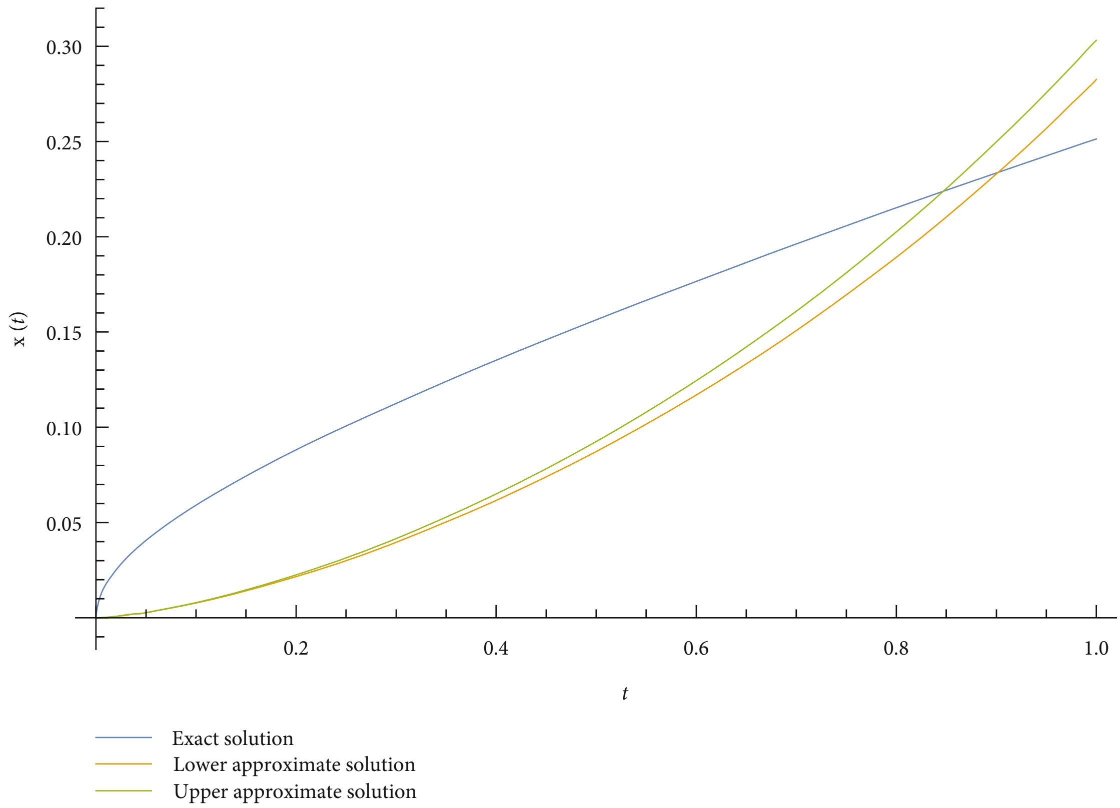

FIGURE 2: Exact solution with approximate solution F^2.

spring of stiffness K to a fixed point. We assume that the small motions of the spring do not disturb the fluid and that the area of the plate, A, is sufficiently large as to produce in the fluid adjacent to the plate the velocity field and stresses developed in the preceding section. For the previous problem, Bagley and Torvik [32] have found the differential equation describing the displacement $x(t)$ of the plate to be

$$m\frac{d^2x}{dt^2} + 2A\sqrt{\nu\rho}D^{3/2}x(t) + Kx(t) = 0, \tag{42}$$

where $D^{3/2} = D^{1/2}x'(t)$ is the R-L fractional derivative of order 3/2, ρ is the fluid density, and ν is the viscosity. The equation above was then known as the Bagley and Torvik equation. Thus, the fractional derivative is established to become visible in the differential equation which depicts the motion of a simple, physical system depending on familiar mechanical and fluid components. Furthermore, its existence may be anticipated in any system distinguished by localized motion in a viscous fluid. Such is the case for oscillations of a polymeric material. Bagley and Torvik in their work [32] believe this accounts for the success of a fractional derivative in modeling these materials.

By replacing R-L fractional derivative by the Caputo fractional derivative with noting that $D^{3/2}D^{1/2}x(t) = D^2x(t)$, we deduce that equation (42) is equivalent to equation (1) with the boundary conditions in (2) and $\alpha = 1/2$, $\beta = 3/2$, $\lambda = 2A\sqrt{\mu\rho}/m$, and $f(t, x(t)) = -Kx(t)/m$. Thus, equation (42) can be rewritten as

$$\frac{d^2x}{dt^2} + \lambda^c D^{3/2}x(t) + \frac{K}{m}x(t) = 0. \tag{43}$$

Since $x(t) \in \mathbb{S} = C([0, 1), \mathbb{R})$, then the $f(t, x(t))$ is continuous and satisfies the Lipschitz condition with $Ł = K/m$.

Let the rigid plate of mass $m = 40$ kg and area $A = 0.2\,\mathrm{m}^2$ immersed in the Newtonian fluid, e.g., water with the values of viscosity and density $\nu = 0.6527$ mPa.s and $\rho = 992.2\,\mathrm{kg/m}^3$ at 40°C and connected by a massless spring of stiffness $K = 10$ N/m to a fixed point which implies that $\lambda = 2A\sqrt{\nu\rho}/m \sim 0.254482$.

By carrying out Mathematica 11 software, it is easy to compute $Q_1 \sim 0.98855$ and $Q_2 \sim 0.818395$ where Q_1 and Q_2 are defined as in (23) and (24), respectively. According to Theorem 12, since $\lambda > 0$, we have to choose $Q = Q_1 \sim 0.98855 = <1$ which implies that there is a unique solution of our problem (43).

Now, we are seeking to compute the exact solution. For this, apply the Laplace transform to the problem (43) to get

$$\begin{aligned}\mathscr{L}\{x(t),s\} &= \frac{1+\lambda s^{-1/2}}{s^2+\lambda s^{3/2}+Ł}x'(0)\\ &= -\frac{1}{\sqrt{s}}\left[\frac{0.189606\sqrt{s}-0.673692}{s-0.884781\sqrt{s}+0.440632}\right.\\ &\quad \left.-\frac{0.189606\sqrt{s}-0.289922}{s+1.13926\sqrt{s}+0.567367}\right]x'(0)\\ &= -\frac{1}{\sqrt{s}}\left[\frac{a_+}{\sqrt{s}-b_+}+\frac{a_-}{\sqrt{s}-b_-}-\frac{c_-}{\sqrt{s}-d_+}-\frac{c_+}{\sqrt{s}-d_-}\right]x'(0),\end{aligned} \tag{44}$$

where

$$\begin{aligned} a_\pm &= 0.0948028 \pm 0.595895i \triangleq a_1 \pm a_2 i,\\ b_\pm &= 0.4423910 \pm 0.494896i \triangleq b_1 \pm b_2 i,\\ c_\pm &= 0.0948028 \pm 0.403712i \triangleq c_1 \pm c_2 i,\\ d_\pm &= -0.569632 \pm 0.492835i \triangleq d_1 \pm d_2 i.\end{aligned} \tag{45}$$

It is known that

$$\mathscr{L}\left\{E_{\alpha,\beta}(\pm a t^\alpha)\right\} = \frac{s^{\alpha-\beta}}{s^\alpha \mp a} \text{ and } E_{1/2,1}(z) = e^{z^2}\operatorname{erfc}(-z), \tag{46}$$

which implies that

$$\begin{aligned} x(t) &= -\left[a_+E_{1/2,1}\left(b_+\sqrt{t}\right)+a_-E_{1/2,1}\left(b_-\sqrt{t}\right)-c_-E_{1/2,1}\left(d_+\sqrt{t}\right)\right.\\ &\quad \left.-c_+E_{1/2,1}\left(d_-\sqrt{t}\right)\right]x'(0)\\ &= -\left[a_+e^{b_+^2 t}\operatorname{erfc}\left(-b_+\sqrt{t}\right)+a_-e^{b_-^2 t}\operatorname{erfc}\left(-b_-\sqrt{t}\right)\right.\\ &\quad \left.-c_-e^{d_+^2 t}\operatorname{erfc}\left(-d_+\sqrt{t}\right)-c_+e^{d_-^2 t}\operatorname{erfc}\left(-d_-\sqrt{t}\right)\right]x'(0)\\ &\triangleq -g(t)x'(0),\end{aligned} \tag{47}$$

where $\operatorname{erfc}(z) = 1-\operatorname{erf}(z)$ is the complementary error function defined as

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt. \tag{48}$$

It is easy to see that

$$\begin{aligned} a_+e^{b_+^2 t}+a_-e^{b_-^2 t} &= e^{\left(b_1^2-b_2^2\right)t}\left[a_1\left(e^{2ib_1b_2t}+e^{-2ib_1b_2t}\right)+ia_2\left(e^{2ib_1b_2t}-e^{-2ib_1b_2t}\right)\right]\\ &= 2e^{\left(b_1^2-b_2^2\right)t}\left[a_1\cos\left(2b_1b_2t\right)+a_2\sin\left(2b_1b_2t\right)\right],\end{aligned}$$

$$\begin{aligned} & a_+e^{b_+^2 t}\operatorname{erf}\left(-b_+\sqrt{t}\right)+a_-e^{b_-^2 t}\operatorname{erf}\left(-b_-\sqrt{t}\right)\\ &= \frac{2}{\sqrt{\pi}}e^{\left(b_1^2-b_2^2\right)t}\left(a_+e^{2ib_1b_2t}\int_0^{-b_+\sqrt{t}}e^{-s^2}ds+a_-e^{-2ib_1b_2t}\int_0^{-b_-\sqrt{t}}e^{-s^2}ds\right)\\ &= -\frac{2\sqrt{t}}{\sqrt{\pi}}e^{\left(b_1^2-b_2^2\right)t}\left(a_+b_+e^{2ib_1b_2t}\int_0^1 e^{-b_+^2ts^2}ds+a_-b_-e^{-2ib_1b_2t}\int_0^1 e^{-b_-^2ts^2}ds\right)\\ &= -\frac{2\sqrt{t}}{\sqrt{\pi}}e^{\left(b_1^2-b_2^2\right)t}\int_0^1 e^{-\left(b_1^2-b_2^2\right)ts^2}\left(a_+b_+e^{2ib_1b_2t\left(1-s^2\right)}+a_-b_-e^{-2ib_1b_2t\left(1-s^2\right)}\right)ds\\ &= -\frac{4\sqrt{t}}{\sqrt{\pi}}e^{\left(b_1^2-b_2^2\right)t}\int_0^1 e^{-\left(b_1^2-b_2^2\right)ts^2}\left(\left(a_1b_1-a_2b_2\right)\cos\left(2b_1b_2t\left(1-s^2\right)\right)\right.\\ &\quad \left.+\left(a_1b_2+a_2b_1\right)\sin\left(2b_1b_2t\left(1-s^2\right)\right)\right)ds,\end{aligned} \tag{49}$$

which lead to $x(t) \in \mathbb{R}$ for all $t \in [0,1]$. Also, it is not difficult to show

$$x(0) = -\left[a_+ + a_- - c_- - c_+\right]x'(0) = -\left[2a_1 - 2c_1\right]x'(0) = 0 \tag{50}$$

and ${}^cD^{12}x(0) = 0$ from the continuity of $x(t)$ and the definition of the Caputo derivative. The last condition in (2) gives

$$x(t) = \frac{\eta}{I^\gamma g(\eta) - g(1) - g(\xi)}g(t). \tag{51}$$

In order to determine the values of minimum and maximum solutions (x_0, y_0), we solve both inequalities in Definition 11 with taking into account the definitions of the two operators F_1 and F_2 as in (27) and (28), respectively, and we find that $x_0 < f(t)$ and $y_0 > f(t)$ for all $t \in [0,1]$ where

$$f(t) = \frac{0.269714t\sqrt{t}}{1+0.287152\sqrt{t}-0.491321t\sqrt{t}+0.125t^2}. \tag{52}$$

In view of the behavior of the function $f(t)$ on the interval $[0,1]$, numerically, we get that it is increasing on $(0,1)$ which yields that $0 = f(0) < f(t) < f(1) \sim 0.292903$. Conclusion, we can take the minimum and maximum solutions $(x_0, y_0) = (0,1)$. Under the assumptions of Theorem 12 and all of which have been fulfilled, there exists a unique solution for the problem (43) in the interval $[0,1]$. With the help of Theorem 5, we can determine the unique solution to this problem as

$$x^* = \lim_{n\longrightarrow\infty}F^n(x_0,y_0) = \lim_{n\longrightarrow\infty}F^n(y_0,x_0), \tag{53}$$

where

$$F^n(x_0,y_0) = F\left(F^{n-1}(x_0,y_0), F^{n-1}(y_0,x_0)\right), n \in \mathbb{N},$$

$$F^0(x_0, y_0) = F(x_0, y_0) = -0.0861455\sqrt{t} + 0.306328t\sqrt{t},$$

$$F^0(y_0, x_0) = F(y_0, x_0) = 0.380497t\sqrt{t} - 0.0375t^2,$$

$$\begin{aligned} F^1(x_0, y_0) = {} & 0.269714t\sqrt{t} - 0.141072t^2 \\ & + 0.0215364t^2\sqrt{t} + 0.159557t^3 \\ & - 0.0428677t^3\sqrt{t}, \end{aligned}$$

$$\begin{aligned} F^1(y_0, x_0) = {} & 0.0247368t + 0.269714t\sqrt{t} - 0.0984762t^2 \\ & + 0.177894t^3 - 0.0614099t^3\sqrt{t} + 0.0046875t^4, \end{aligned}$$

$$\begin{aligned} F^2(x_0, y_0) = {} & 0.262611t\sqrt{t} - 0.0774489t^2 \\ & + 0.0312966t^2\sqrt{t} + 0.132516t^3 \\ & - 0.14891t^3\sqrt{t} + 0.0432208t^4 \\ & + 0.0765936t^4\sqrt{t} - 0.0432694t^5 \\ & + 0.00593055t^5\sqrt{t}, \end{aligned}$$

$$\begin{aligned} F^2(y_0, x_0) = {} & 0.269714t\sqrt{t} - 0.0774489t^2 \\ & + 0.0496437t^2\sqrt{t} + 0.12324t^3 \\ & - 0.133113t^3\sqrt{t} + 0.0272475t^4 \\ & + 0.0851651t^4\sqrt{t} - 0.0501457t^5 \\ & + 0.00940721t^5\sqrt{t} - 0.000585938t^6. \end{aligned} \quad (54)$$

In Figures 1 and 2, we graph the exact solution with blue color, lower approximate solution with yellow color, and upper approximate solution with green color. We note that the lower and upper solutions take the same behavior slightly different and intersect the exact solution at $t = 0.95$ and $t = 0.75$ in Figure 1 and $t = 0.9$ and $t = 0.8$ in Figure 2, respectively. Therefore, we expect that they intersect at the same point in the higher order.

5. Conclusion

Here, some transactions are presented with the uniqueness of solution for four-point and strip generalized Langevin equation that included two generalized orders in distinct intervals. We solve the problem with the properties of the fractional calculus and coupled immutable point type. It turned out that there are two cases to apply our theorems. We took each case into consideration and gave an illustrative example for each one. The fractional differential equation, due to Bagley and Torvik which describes the motion of the rigid plate immersed in a Newtonian fluid of infinite extent and connected by a massless spring of stiffness to a fixed point, is taken as an application of our results. It is shown in Figures 1 and 2 that there is a small difference between the exact and approximate solutions.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this paper. All authors have read and agreed to the final version of the manuscript.

Acknowledgments

This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia, under grant no. KEP-PhD-60-130-42. The authors, therefore, acknowledge with thanks DSR for their technical and financial support.

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