Hindawi
Advances in Mathematical Physics
Volume 2021, Article ID 5555930, 7 pages
https://doi.org/10.1155/2021/5555930

Research Article

A New Result for a Blow-up of Solutions to a Logarithmic Flexible Structure with Second Sound

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Received 21 January 2021; Revised 2 February 2021; Accepted 4 February 2021; Published 11 February 2021

Academic Editor: Kamyar Hosseini

This paper is concerned with a problem of a logarithmic nonuniform flexible structure with time delay, where the heat flux is given by Cattaneo's law. We show that the energy of any weak solution blows up infinite time if the initial energy is negative.

1. Introduction

In this work, we consider the vibrations of an inhomogeneous flexible structure system with a constant internal delay and logarithmic nonlinear source term:

$$\begin{cases} m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{tx})_x + \eta\theta_x + \mu u_t(x, t-\tau_0) = u|u|^{p-2} \ln |u|^{\gamma}, & x \in (0, L), t > 0, \\ \theta_t + kq_x + \eta u_{tx} = 0 & x \in (0, L), t > 0, \\ \tau q_t + \beta q + k\theta_x = 0 & x \in (0, L), t > 0, \end{cases} \tag{1}$$

with boundary conditions

$$u(0, t) = u(L, t) = 0 \,;\, \theta(0, t) = \theta(L, t) = 0, t \ge 0, \tag{2}$$

and initial conditions

$$u(x, 0) = u_0(x), u_t(x, 0) = u_1(x) \,;\, \theta(x, 0) = \theta_0(x) \,;\, q(x, 0) = q_0(x), x \in [0, L], \tag{3}$$

where $u(x, t)$ is the displacement of a particle at position $x \in [0, L]$, and the time $t > 0$. $\eta > 0$ is the coupling constant depending on the heating effect,$p \ge 2$,γ, β, and k are positive constants, and μ is a real number. $\tau > 0$ is the relaxation time describing the time lag in the response for the temperature, and $\tau_0 > 0$ represents the time delay in particular if $\tau = 0$(1.1) reduces to the viscothermoelastic system with delay, in which the heat flux is given by Fourier's law instead of Cattaneo's law, where $q = q(x, t)$ is the heat flux, and $m(x)$, $\delta(x)$, and $p(x)$ are responsible for the inhomogeneous structure of the beam and, respectively, denote mass per unit length of structure, coefficient of internal material damping (viscoelastic property), and a positive function related to the stress acting on the body at a point x. The model of heat condition, originally due to Cattaneo, is of hyperbolic type. We recall the assumptions of $m(x)$, $\delta(x)$, and $p(x)$ in [1, 2] such that

$$m, \delta, p \in W^{1,\infty}(0, L), m(x), \delta(x) \text{ and } p(x) > 0, \forall x \in [0, L]. \tag{4}$$

In these kinds of problems, Gorain [3] in 2013 has established uniform exponential stability of the problem

$$m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{tx})_x = f(x), on\ (0, L) \times \mathbb{R}^+, \tag{5}$$

which describes the vibrations of an inhomogeneous flexible structure with an exterior disturbing force. More recently, Misra et al. [4] showed the exponential stability of the vibrations of a inhomogeneous flexible structure with thermal effect governed by the Fourier law.

$$\begin{aligned} m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{tx})_x - k\theta_x = f(x), \\ \theta_t - \theta_{xx} - ku_{xt} = 0. \end{aligned} \tag{6}$$

In addition, we can cite other works in the same form like the system in [5]; Racke studied the exponential stability in linear and nonlinear 1d of thermoelasticity system with second sound given by

$$\begin{cases} m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{tx})_x - k\theta_x = 0, on(0,L) \times \mathbb{R}^+ \\ \theta_t + kq_x + \eta u_{tx} = 0, on(0,L) \times \mathbb{R}^+ \\ \tau q_t + \beta q + k\theta_x = 0, on(0,L) \times \mathbb{R}^+, \end{cases} \tag{7}$$

Now for the multidimensional system, Messaoudi in [6] established a local existence and a blow-up result for a multidimensional nonlinear system of thermoelasticity with second sound (see in this regard Refs. [7–10]); for the same problem above, Alves et al. proved that system (7) is polynomial decay (see [1]), with boundary and initial conditions:

$$\begin{aligned} u(0,t) = u(L,t) = 0\,; \theta(0,t) = \theta(L,t) = 0, t \ge 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x)\,; \\ \theta(x,0) = \theta_0(x)\,; q(x,0) = q_0(x), x \in [0,L]. \end{aligned} \tag{8}$$

We know that the dynamic systems with delay terms have become a significant examination subject in differential condition since the 1970s of the only remaining century. The delay effect that is similar to memory processes is important in the research of applied mathematics such as physics, non-instant transmission phenomena, and biological motivation; model (7) is related to the following problem with delay terms:

$$\begin{cases} m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{tx})_x + \eta\theta_x + \mu u_t(x, t-\tau_0) = 0 & x \in (0,L), t > 0, \\ \theta_t + kq_x + \eta u_{tx} = 0 & x \in (0,L), t > 0, \\ \tau q_t + \beta q + k\theta_x = 0 & x \in (0,L), t > 0, \\ u(0,t) = u(L,t) = 0\,; \theta(0,t) = \theta(L,t) = 0, t \ge 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x)\,; \\ \theta(x,0) = \theta_0(x)\,; q(x,0) = q_0(x), x \in [0,L]. \end{cases} \tag{9}$$

The authors prove that the system (9) is well posed and exponential decay under a small condition on time delay (see [2]). Now in the presence of source term, the system (9) becomes the system studied in this work with a logarithmic source term; this type of problems is encountered in many branches of physics such as nuclear physics, optics, and geophysics. It is well known, from the quantum field theory, that such kind of logarithmic nonlinearity appears naturally in inflation cosmology and in supersymmetric field theories (see [11–13]).

This work is organized as follows: In "Statement of Problem," we talk briefly about the local existence of the systems (1), (2), and (3), and we define some space and theorem used. In "Blow-up of Solution," the blow-up result is proved.

2. Statement of Problem

Let us introduce the function

$$z(x,\rho,t) = u_t(x, t-\rho\tau_0), x \in (0,L), \rho \in (0,1), t > 0. \tag{10}$$

Thus, we have

$$\tau_0 z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, x \in (0,L), \rho \in (0,1), t > 0. \tag{11}$$

Then, problems (1)–(3) are equivalent to

$$\begin{cases} m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{tx})_x + \eta\theta_x & \\ +\mu z(x,1,t) = u|u|^{p-2}\ln|u|^\gamma, & x \in (0,L), t > 0, \\ \tau_0 z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 & x \in (0,L), \rho \in (0,1), t > 0, \\ \theta_t + kq_x + \eta u_{tx} = 0 & x \in (0,L), t > 0, \\ \tau q_t + \beta q + k\theta_x = 0 & x \in (0,L), t > 0, \end{cases} \tag{12}$$

$$\begin{cases} u(0,t) = u(L,t) = 0\,; \theta(0,t) = \theta(L,t) = 0, t \ge 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x)\,; \theta(x,0) = \theta_0(x), x \in [0,L] \\ q(x,0) = q_0(x), x \in [0,L] \\ z(x,0,t) = u_t(x,t), x \in (0,L), t > 0, \\ z(x,\rho,0) = f_0(x, -\rho\tau_0), x \in (0,L), \rho \in (0,1). \end{cases} \tag{13}$$

We first state a local existence theorem that can be established by combining the arguments of related works [10,6].

Let $v = u_t$ and denote by

$$\Phi = (u, v, \theta, q, z)^T, \Phi(0) = \Phi_0 = (u_0, u_1, \theta_0, q_0, f_0)^T. \tag{14}$$

The state space of Φ is the Hilbert space

$$= H_0^1(0,L) \times L^2(0,L) \times L_*^2(0,L) \times L^2((0,1)\times(0,L)). \tag{15}$$

Theorem 1. *Assume that*

$$2 < p \le \frac{2n}{n-2}, if\ n \ge 3. \tag{16}$$

Then, for every $\Phi_0 \in$, there exists a unique local solution in the class $\Phi \in C([0,T],)$.

3. Blow-up of Solution

In this section, we prove that the solutions for the problems (12)–(13) blow up in a finite time when the initial energy is negative. We use the improved method of Salim and Messaoudi [6]. We define the energy associated with problems (12)–(13) by

$$\begin{aligned} E(t) &= \frac{1}{2}\left(\|m(x)\|_\infty \|u_t(t)\|_2^2\right) + \frac{1}{2}\left(\|p(x)\|_\infty \|u_x\|_2^2\right) \\ &\quad + \frac{\tau}{2}\|q\|_2^2 + \frac{1}{2}\|\theta\|_2^2 + \frac{\tau_0|\mu|}{2}\int_0^1 \|z(x,\rho,t)\|^2 d\rho \\ &\quad + \frac{\gamma}{p^2}\|u\|_p^p - \frac{1}{p}\int_0^L |u|^p \ln |u|^\gamma dx. \end{aligned} \tag{17}$$

Lemma 2. *Suppose that*

$$2 < p \le \frac{2n}{n-2}, n \ge 3. \tag{18}$$

Then, there exists a positive constant $C > 0$ depending on $[0.L]$ only, such that

$$\left[\int_0^L |u|^p \ln |u|^\gamma dx\right]^{\frac{s}{p}} \le C\left[\int_0^L |u|^p \ln |u|^\gamma dx + \|u_x\|_2^2\right], \tag{19}$$

for any $u \in H_0^1(0,L)$ and $2 \le s \le p$, provided that $\int_0^L |u|^p \ln |u|^\gamma dx \ge 0$.

Proof. If $\int_0^L |u|^p \ln |u|^\gamma dx > 1$, then

$$\left[\int_0^L |u|^p \ln |u|^\gamma dx\right]^{\frac{s}{p}} \le \int_0^L |u|^p \ln |u|^\gamma dx. \tag{20}$$

If $\int_0^L |u|^p \ln |u|^\gamma dx \le 1$, then we set

$$\Gamma_1 = \{x \in [0,L] \mid |u| > 1\}, \tag{21}$$

and, for any $\beta \le 2$, we have

$$\begin{aligned} \left[\int_0^L |u|^p \ln |u|^\gamma dx\right]^{\frac{s}{p}} &\le \left[\int_0^L |u|^p \ln |u|^\gamma dx\right]^{\frac{\beta}{p}} \le \left[\int_{\Gamma_1} |u|^p \ln |u|^\gamma dx\right]^{\frac{\beta}{p}} \\ &\le \left[\int_{\Gamma_1} |u|^{p+1} dx\right]^{\frac{\beta}{p}} \le \left[\int_0^L |u|^{p+1} dx\right]^{\frac{\beta}{p}} = \|u\|_{p+1}^{\frac{\beta(p+1)}{p}}. \end{aligned} \tag{22}$$

We choose $\beta = 2p/(p+1) < 2$ to get

$$\left[\int_0^L |u|^p \ln |u|^\gamma dx\right]^{\frac{s}{p}} \le \|u\|_{p+1}^2 \le C\|u_x\|_2^2. \tag{23}$$

Combining (20) and (23), the result was obtained.

Lemma 3. *There exists a positive constant $C > 0$ depending on $[0,L]$ only, such that*

$$\|u\|_p^p \le C\left[\int_0^L |u|^p \ln |u|^\gamma dx + \|u_x\|_2^2\right], \tag{24}$$

for any $u \in H_0^1(0,L)$, provided that $\int_0^L |u|^p \ln |u|^\gamma dx \ge 0$.

Proof. We set

$$\Gamma_+ = \{x \in [0,L] \mid |u| > e\} \text{ and } \Gamma_- = \{x \in [0,L] \mid |u| \le e\}, \tag{25}$$

thus

$$\begin{aligned} \|u\|_p^p &= \int_{\Gamma_+} |u|^p dx + \int_{\Gamma_-} |u|^p dx \le \int_{\Gamma_+} |u|^p \ln |u|^\gamma dx + \int_{\Gamma_-} e^p \left|\frac{u}{e}\right|^p dx \\ &\le \int_{\Gamma_+} |u|^p \ln |u|^\gamma dx + e^p \int_{\Gamma_-} \left|\frac{u}{e}\right|^2 dx \\ &\le \int_0^L |u|^p \ln |u|^\gamma dx + e^{p-2}\int_0^L |u|^2 dx \\ &\le C\left\{\int_0^L |u|^p \ln |u|^\gamma dx + \|u_x\|_2^2\right\}. \end{aligned} \tag{26}$$

By using the inequalities $\|u\|_2^2 \le C\|u\|_p^2 \le C(\|u\|_p^p)^{2/p}$, we have the following corollary.

Corollary 4. *There exists a positive constant $C > 0$ depending on $[0,L]$ only, such that*

$$\|u\|_2^2 \le C\left[\left(\int_0^L |u|^p \ln |u|^\gamma dx\right)^{\frac{2}{p}} + \|u_x\|_2^{\frac{4}{p}}\right]. \tag{27}$$

provided that $\int_0^L |u|^p \ln |u|^\gamma dx \ge 0$.

Lemma 5. *There exists a positive constant $C > 0$ depending on $[0,L]$ only, such that*

$$\|u\|_p^s \le C\left[\|u\|_p^p + \|u_x\|_2^2\right], \tag{28}$$

for any $u \in H_0^1(0,L)$ and $2 \le s \le p$.

Proof. If $\|u\|_p \ge 1$, then

$$\|u\|_p^s \le \|u\|_p^p. \tag{29}$$

If $\|u\|_p \le 1$, then $\|u\|_p^s \le \|u\|_p^2$. Using the Sobolev embedding theorems, we have

$$\|u\|_p^s \le \|u\|_p^2 \le C\|u_x\|_2^2. \tag{30}$$

Now we are ready to state and prove our main result. For this purpose, we define

$$H(t) = -E(t) = -\frac{1}{2}\left(\|m(x)\|_\infty \|u_t(t)\|_2^2\right) - \frac{1}{2}\left(\|p(x)\|_\infty \|u_x\|_2^2\right) - \frac{\tau}{2}\|q\|_2^2 - \frac{1}{2}\|\theta\|_2^2 - \frac{\tau_0|\mu|}{2}\int_0^1 \|z(x,\rho,t)\|^2 d\rho - \frac{\gamma}{p^2}\|u\|_p^p + \frac{1}{p}\int_0^L |u|^p \ln |u|^\gamma dx. \tag{31}$$

Corollary 6. *Assume that (18) holds. Then*

$$\|u\|_p^s \le C\left\{\left(1 - \frac{\gamma}{p\|p(x)\|_\infty}\right)\|u\|_p^p - \left(\frac{2}{\|p(x)\|_\infty}\right)H(t) - \left(\frac{\|m(x)\|_\infty}{\|p(x)\|_\infty}\right)\|u_t(t)\|_2^2 - \left(\frac{\tau}{\|p(x)\|_\infty}\right)\|q\|_2^2 - \left(\frac{1}{\|p(x)\|_\infty}\right)\|\theta\|_2^2 - \frac{\tau_0|\mu|}{\|p(x)\|_\infty}\int_0^1 \|z(x,\rho,t)\|^2 d\rho + \frac{2}{p\|p(x)\|_\infty}\int_0^L |u|^p \ln |u|^\gamma dx\right\}, \tag{32}$$

for any $u \in (H_0^1(0,L))^n$ and $2 \le s \le p$.

Theorem 7. *Assume that (18) holds. Assume further that*

$$E(0) = \frac{1}{2}\left(\|m(x)\|_\infty \|u_1(t)\|_2^2\right) + \frac{1}{2}\left(\|p(x)\|_\infty \|\nabla u_0\|_2^2\right) + \frac{\tau}{2}\|q_0\|_2^2 + \frac{1}{2}\|\theta_0\|_2^2 + \frac{\tau_0|\mu|}{2}\int_0^L\int_0^1 |f_0(x,-\rho\tau_0)|^2 d\rho dx + \frac{\gamma}{p^2}\|u_0\|_p^p - \frac{1}{p}\int_0^L |u_0|^p \ln |u_0|^\gamma dx < 0. \tag{33}$$

Then, the solution of (12) blows up in finite time.

Proof. we have

$$E(t) \le E(0) < 0, \tag{34}$$

and

$$H'(t) = -E'(t) = 2\left(\|\delta(x)\|_\infty \|u_{xt}(t)\|_2^2\right) + \beta\|q\|_2^2 + |\mu|\int_0^L |z(x,1,t)|^2 dx. \tag{35}$$

Hence

$$H'(t) \ge C_0\left\{\left(\|\delta(x)\|_\infty \|u_{xt}(t)\|_2^2\right) + |\mu|\int_0^L z^2(x,1,t)dx\right\} \ge 0; \forall t \in [0,T). \tag{36}$$

Consequently, we get

$$0 < H(0) \le H(t) \le \int_0^L |u|^p \ln |u|^\gamma dx; \forall t \in [0,T), \tag{37}$$

by virtue of (17) and (31). We then introduce

$$L(t) = H^{1-\alpha}(t) + \varepsilon\int_0^L \left[m(x)u_t(t)u(t) + 4\delta(x)|u_x|^2\right]dx + \varepsilon\int_0^L \frac{n\tau}{k} uq dx, \tag{38}$$

where $\varepsilon > 0$ to be specified later and

$$\frac{2(p-2)}{p^2} < \alpha < \frac{p-2}{2p} < 1. \tag{39}$$

A direct differentiation of $L(t)$ gives

$$L'(t) = (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\int_0^L m(x)|u_t|^2 dx + \varepsilon\frac{\eta\tau}{k}\int_0^L q u_t(t)dx - \varepsilon\int_0^L p(x)|u_x|^2 dx + 2\varepsilon\eta\int_0^L \theta u_x dx - \varepsilon\int_0^L \mu z(x,1,t)u dx + \varepsilon\int_0^L |u|^p \ln |u|^\gamma - \varepsilon\frac{\eta\beta}{k}\int_0^L q u dx, \tag{40}$$

using the inequality of young

$$2\varepsilon\eta\int_0^L \theta u_x dx \ge -\varepsilon\eta\|\theta\|_2^2 - \varepsilon\eta\|u_x\|_2^2, \tag{41}$$

$$-\varepsilon\int_0^L p(x)|u_x|^2 dx \ge -\varepsilon\|p(x)\|_\infty \|u_x\|_2^2, \tag{42}$$

$$\varepsilon\int_0^L m(x)|u_t|^2 dx \ge \varepsilon\|m(x)\|_\infty \|u_t\|_2^2, \tag{43}$$

$$\varepsilon\frac{\eta\tau}{k}\int_0^L q u_t(t)dx \ge -\varepsilon\frac{\eta\tau}{2k}\|u_t\|_2^2 - \varepsilon\frac{\eta\tau}{2k}\|q\|_2^2, \tag{44}$$

and

$$-\varepsilon\int_0^L \mu z(x,1,t)u dx \ge -\varepsilon|\mu|\left\{\frac{\xi_1}{2}\int_0^L |z(x,1,t)|^2 dx + \frac{1}{2\xi_1}\|u\|_2^2\right\}, \forall \xi_1 > 0, \tag{45}$$

$$-\varepsilon\frac{\eta\beta}{k}\int_0^L q u dx \ge \varepsilon\frac{\eta\beta}{k}\left\{\frac{\xi_2}{2}\|q\|_2^2 + \frac{1}{2\xi_2}\|u\|_2^2\right\}, \forall \xi_2 > 0. \tag{46}$$

Substituting (41), (42), (43), (44), (45), and (46) in (40), we get

$$\begin{aligned} L'(t) \geq{}& (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\left\{\|m(x)\|_\infty - \frac{\eta\tau}{2k}\right\}\|u_t\|_2^2 \\ &- \varepsilon\left\{\|p(x)\|_\infty + \eta\right\}\|u_x\|_2^2 - \varepsilon\eta\|\theta\|_2^2 \\ &- \varepsilon\left\{\frac{\tau\eta + \beta\eta\xi_2}{2k}\right\}\|q\|_2^2 + \varepsilon\int_0^L |u|^p \ln|u|^\gamma dx \\ &- \varepsilon|\mu|\frac{\xi_1}{2}\int_0^L |z(x,1,t)|^2 dx - \varepsilon\|u\|_2^2\left\{\frac{|\mu|}{2\xi_1} + \frac{\eta\beta}{2\xi_2 k}\right\}. \end{aligned} \tag{47}$$

We obtain from (35) and (47) the following:

$$\begin{aligned} L'(t) \geq{}& \left\{(1-\alpha)H^{-\alpha}(t) - \varepsilon\left(\frac{k\xi_1 + \eta\xi_2}{2k}\right)\right\}H'(t) \\ &+ \varepsilon\left\{\|m(x)\|_\infty - \frac{\eta\tau}{2k}\right\}\|u_t\|_2^2 - \varepsilon\left\{\|p(x)\|_\infty + \eta\right\}\|u_x\|_2^2 \\ &- \varepsilon\eta\|\theta\|_2^2 - \varepsilon\frac{\tau\eta}{2k}\|q\|_2^2 + \varepsilon\int_0^L |u|^p \ln|u|^\gamma dx \\ &- \varepsilon\left\{\frac{|\mu|k}{2\xi_1 k} + \frac{\eta\beta}{2\xi_2 k}\right\}\|u\|_2^2. \end{aligned} \tag{48}$$

We also set $\xi_1 = \xi_2 = H^{-\alpha}(t)$; hence, (48) gives

$$\begin{aligned} L'(t) \geq{}& \{(1-\alpha) - \varepsilon C\}H^{-\alpha}(t)H'(t) + \varepsilon\left\{\|m(x)\|_\infty - \frac{\eta\tau}{2k}\right\}\|u_t\|_2^2 \\ &- \varepsilon\left\{\|p(x)\|_\infty + \eta\right\}\|u_x\|_2^2 - \varepsilon\eta\|\theta\|_2^2 - \varepsilon\frac{\tau\eta}{2k}\|q\|_2^2 \\ &+ \varepsilon\int_0^L |u|^p \ln|u|^\gamma dx - \varepsilon\frac{M}{2k}H^\alpha(t)\|u\|_2^2, \end{aligned} \tag{49}$$

where C and M are strictly positive constants depending only on $k, \eta, \beta, |\mu|$.

For $0 < a < 1$, we have

$$\begin{aligned} L'(t) \geq{}& \{(1-\alpha) - \varepsilon C\}H^{-\alpha}(t)H'(t) \\ &+ \varepsilon\left\{\|m(x)\|_\infty\left(1 + \frac{p}{2}(1-a)\right) - \frac{\eta\tau}{2k}\right\}\|u_t\|_2^2 \\ &+ \varepsilon\left\{\|p(x)\|_\infty\left(\frac{p}{2}(1-a) - 1\right) + \eta\right\}\|u_x\|_2^2 \\ &+ \varepsilon\left\{-\eta + \frac{p\varepsilon(1-a)}{2}\right\}\|\theta\|_2^2 + \varepsilon a\int_0^L |u|^p \ln|u|^\gamma dx \\ &+ \varepsilon\left\{-\frac{\tau\eta}{2k} + \frac{p\tau(1-a)}{2}\right\}\|q\|_2^2 + \frac{\gamma\varepsilon(1-a)}{2}\|u\|_p^p \\ &+ \varepsilon\frac{\tau_0 p(1-a)}{2}|\mu|\int_0^L\int_0^1 |z(x,\rho,t)|^2 d\rho dx \\ &- \varepsilon\frac{M}{2k}H^\alpha(t)\|u\|_2^2 + p\varepsilon(1-a)H(t). \end{aligned} \tag{50}$$

Using (27), (37) and Young's inequality, we find

$$\begin{aligned} H^\alpha(t)\|u\|_2^2 &\leq \left(\int_0^L |u|^p \ln|u|^\gamma dx\right)^\alpha \|u\|_2^2 \\ &\leq C\left[\left(\int_0^L |u|^p \ln|u|^\gamma dx\right)^{\alpha+\frac{2}{p}} + \left(\int_0^L |u|^p \ln|u|^\gamma dx\right)^\alpha \|u_x\|_2^{\frac{4}{p}}\right] \\ &\leq C\left[\left(\int_0^L |u|^p \ln|u|^\gamma dx\right)^{\frac{(\alpha p+2)}{p}} + \|u_x\|_2^2 + \left(\int_0^L |u|^p \ln|u|^\gamma dx\right)^{\frac{\alpha p}{p-2}}\right]. \end{aligned} \tag{51}$$

Exploiting (39), we have

$$2 < \alpha p + 2 \leq p \text{ and } 2 < \frac{\alpha p^2}{p-2} \leq p. \tag{52}$$

Thus, lemma 1 yields

$$H^\alpha(t)\|u\|_2^2 \leq C\left\{\int_0^L |u|^p \ln|u|^\gamma dx + \|u_x\|_2^2\right\}. \tag{53}$$

Combining (50) and (53), we obtain

$$\begin{aligned} L'(t) \geq{}& \{(1-\alpha) - \varepsilon C\}H^{-\alpha}(t)H'(t) \\ &+ \varepsilon\left\{\|m(x)\|_\infty\left(1 + \frac{p}{2}(1-a)\right) - \frac{\eta\tau}{2k}\right\}\|u_t\|_2^2 \\ &+ \varepsilon\left\{\|p(x)\|_\infty\left(\frac{p}{2}(1-a) - 1\right) + \eta - C\frac{M}{2k}\right\}\|u_x\|_2^2 \\ &+ \varepsilon\left\{-\eta + \frac{p\varepsilon(1-a)}{2}\right\}\|\theta\|_2^2 \\ &+ \varepsilon\left\{a - C\frac{M}{2k}\right\}\int_0^L |u|^p \ln|u|^\gamma dx \\ &+ \varepsilon\left\{-\frac{\tau\eta}{2k} + \frac{p\tau(1-a)}{2}\right\}\|q\|_2^2 + \frac{\gamma\varepsilon(1-a)}{2}\|u\|_p^p \\ &+ \varepsilon\frac{\tau_0 p(1-a)}{2}|\mu|\int_0^L\int_0^1 |z(x,\rho,t)|^2 d\rho dx + p\varepsilon(1-a)H(t). \end{aligned} \tag{54}$$

At this point, we choose $a > 0$ so small that

$$\begin{aligned} &-\eta + \frac{p\varepsilon(1-a)}{2} > 0, \\ &\left(\frac{p}{2}(1-a) - 1\right) > 0, \\ &\frac{\tau_0 p(1-a)}{2} > 0, \end{aligned} \tag{55}$$

and k so large that

$$\begin{aligned}
&\|p(x)\|_\infty\left(\frac{p}{2}(1-a)-1\right)+\eta-C\frac{M}{2k}>0,\\
&a-C\frac{M}{2k}>0,\\
&\|m(x)\|_\infty\left(1+\frac{p}{2}(1-a)\right)-\frac{\eta\tau}{2k}>0,\\
&-\frac{\tau\eta}{2k}+\frac{p\tau(1-a)}{2}>0.
\end{aligned} \tag{56}$$

Once C and a are fixed, we pick ε so small so that

$$(1-\alpha)-\varepsilon C>0. \tag{57}$$

Hence, (54) becomes

$$\begin{aligned}
L'(t)\geq{}&\{(1-\alpha)-\varepsilon C\}H^{-\alpha}(t)H'(t)+\varepsilon A_1\|u_t\|_2^2+\varepsilon A_2\|u_x\|_2^2\\
&+\varepsilon A_3\|\theta\|_2^2+\varepsilon A_4\|q\|_2^2+\varepsilon\left\{a-C\frac{M}{2k}\right\}\int_0^L|u|^p\ln|u|^\gamma dx\\
&+\varepsilon\frac{\tau_0 p(1-a)}{2}|\mu|\int_0^L\int_0^1|z(x,\rho,t)|^2d\rho dx\\
&+\frac{\gamma\varepsilon(1-a)}{2}\|u\|_p^p+p\varepsilon(1-a)H(t),
\end{aligned} \tag{58}$$

where A_1-A_4 are strictly positive constants depending only on p,τ,η,k,a.

Thus, for some $A_0>0$, estimate (58) becomes

$$\begin{aligned}
L'(t)\geq A_0\Big\{&H(t)+\|u_t\|_2^2+\|u_x\|_2^2+\|u\|_p^p\|q\|_2^2+\|\theta\|_2^2\\
&+\int_0^L|u|^p\ln|u|^\gamma dx+\int_0^L\int_0^1|z(x,\rho,t)|^2d\rho dx\Big\},
\end{aligned} \tag{59}$$

and

$$L(t)\geq L(0)>0,\forall t\geq 0. \tag{60}$$

Next, using Hôlder's inequality and the embedding $\|u\|_2\leq C\|u\|_p$, we have

$$\left|\int_0^L m(x)uu_t dx\right|\leq\|m(x)\|_\infty\|u\|_2\|u_t\|_2\leq C\|u\|_2\|u_t\|_2, \tag{61}$$

and exploiting Young's inequality, we obtain

$$\left|\int_0^L m(x)uu_t dx\right|^{\frac{1}{1-\alpha}}\leq C\left\{\|u\|_p^{\frac{r}{1-\alpha}}+\|u_t\|_2^{\frac{r'}{1-\alpha}}\right\},\ \text{For}\ \frac{1}{r}+\frac{1}{r'}=1. \tag{62}$$

To be able to use Lemma 5, we take $r'=2(1-\alpha)$ which gives $r/1-\alpha=2/1-2\alpha\leq p$.

Therefore, for $s=2/1-2\alpha$, estimate (62) yields

$$\left|\int_0^L m(x)uu_t dx\right|^{\frac{1}{1-\alpha}}\leq C\left(\|u\|_p^s+\|u_t\|_2^2\right). \tag{63}$$

Hence, Lemma 5 gives

$$\left|\int_0^L m(x)uu_t dx\right|^{\frac{1}{1-\alpha}}\leq C_1\left(\|u\|_p^p+\|u_t\|_2^2+\|u_x\|_2^2\right),\forall C_1>0, \tag{64}$$

and with the same way, we get

$$\left|\varepsilon\int_0^L\frac{n\tau}{k}uqdx\right|^{\frac{1}{1-\alpha}}\leq C_2\left(\|u\|_p^p+\|q\|_2^2\right),\forall C_2>0, \tag{65}$$

$$\left|\varepsilon\int_0^L 4\delta(x)|u_x|^2dx\right|^{\frac{1}{1-\alpha}}\leq C_3\|u_x\|_2^2,\forall C_3>0. \tag{66}$$

From (64), (65), and (66) we obtain

$$L^{\frac{1}{1-\alpha}}(t)\leq C\left\{H(t)+\|u\|_p^p+\|q\|_2^2+\|u_x\|_2^2+\|u_t\|_2^2\right\};\forall t\geq 0,\forall C>0. \tag{67}$$

Combining (67) and (59), we arrive at

$$L'(t)\geq a_0L^{\frac{1}{1-\alpha}}(t),\forall t\geq 0, \tag{68}$$

where a_0 is a positive constant depending only on A_0 and C.

A simple integration of (68) over $(0,t)$ yields

$$L^{\frac{\alpha}{1-\alpha}}(t)\geq\frac{1}{L^{-\alpha/1-\alpha}(0)-\alpha a_0 t/(1-\alpha)}. \tag{69}$$

Therefore, $L(t)$ blows up in time

$$T^*\leq\frac{1-\alpha}{\alpha a_0L^{\alpha/1-\alpha}(0)}. \tag{70}$$

The proof is completed.

4. Conclusion

In this work, we are interested with a problem of a logarithmic nonuniform flexible structure with time delay, where the heat flux is given by Cattaneo's law. We show that the energy of any weak solution blows up infinite time if the initial energy is negative. The delay effect that is similar to memory processes is important in the research of applied mathematics such as physics, noninstant transmission phenomena, and biological motivation. In the future work, we will try to study the local existence for this problem with respect to some proposal conditions.

Data Availability

No data were used to support the study.

Conflicts of Interest

This work does not have any conflicts of interest.

Acknowledgments

The authors are grateful to the anonymous referees for the careful reading and their important observations/suggestions for the sake of improving this paper.

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